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## Population Growth and Human Capital: A Welfarist Approach

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# Population Growth and human capital: a Welfarist Approach

Abstract

*In this paper we investigate the relationship between economic and population growth in an endogenous growth model driven by human capital accumulation à la Lucas (1988). Since we allow for endogenous population growth, we adopt the population criterion Relative Critical Level Utilitarianism (an extension of Critical Level Utilitarianism, Blackorby et al. 1995) which allows axiomatically founded welfare orderings under variable population. Under this extension the Critical Level Utility is dependent on parents' wellbeing. In this scenario we investigate the equilibrium relation between economic growth and population growth as functions of the underlying parameters and we provide the conditions for the economic take-off to occur.*

**JEL Classification:** D6, J11, O41.

**Keywords:** population growth, endogenous growth, human capital, critical level utilitarianism.

## 1. Introduction

Endogenous growth models have been flourishing over the last three decades. Starting from the pioneering work by Romer (1986) such a strand of literature has provided new insights on the relationship between human capital accumulation, technological progress and economic growth.

Summarizing, these contributions have clarified that long-run per capita growth, in the absence of exogenous technological progress, can only be achieved if returns to capital are constant asymptotically (see Barro and Sala-i-Martin 2004, ch. 5). For doing this, Romer (1986) introduces externalities deriving from existing capital (spillovers as "learning by doing"). On the other hand, Lucas (1988), building on a framework developed by Uzawa (1964) showed that decreasing returns to capital could be avoided by adopting a broad view of capital itself that entails human capital as well. Under the Uzawa-Lucas model, human capital accumulation is in fact the engine of growth that can avoid diminishing returns to capital.

Following the works by Romer and Lucas, several scholars have contributed to the literature on endogenous growth. For example, Bond et al (1996), Mino (1996) and Ventura (1997) extend Romer (1986) by allowing for neoclassical production function of human capital. In a subsequent work, Romer (1990) himself applied the concept of nonrivalty to "ideas" or "discoveries" that can enhance production efficiency and technological progress, and obtained increasing returns in production and thus sustained per-capita income growth.

Nevertheless we argue that the role of endogenous population growth has been somehow overlooked so far[1].

To summarize, in Lucas (1988) model and in subsequent works on human capital accumulation population growth rate is exogenous. Moreover, some of the other early models exhibit scale effect, in the sense that economies with larger populations grow faster (see, for example Romer 1990).

Jones (1995, 1999), by questioning such an outcome, works out a variant of the original framework whereby sustained and stable equilibrium growth of per capita income is only possible in an economy with exogenously growing population[2].

[1] For a recent review on this topic, see Prettner and Prskawetz (2010).

[2] In fact, Jones (2003) writes: "More people means more Isaac Newtons and therefore more ideas." (p. 12). Jones' influential works have originated the so-called "semi-endogenous" growth literature, given that exogenous population growth is the ultimate engine for sustained growth and the latter cannot be influenced by taxes or public policies. Other authors such as Kortum (1997) and Prettner (2009) have

However, once other factors such as education, quality of institutions, country size and openness to trade have been taken into account, there seems to be little or no evidence at all, in the postwar data or from the historical data of the past 200 years, that faster population growth leads to a higher equilibrium growth rate (see Acemoglu 2009, p. 448, Bloom et. al. 2003, p. 17).

In fact, other models such as the one provided by Dalgaard and Kreiner (2001), entail a negative correlation between population growth and economic growth, while Strulik (2005) proposes a model in which the effect of population growth is ambiguous, depending on the degree of parents altruism towards future generations. Analogously, Bucci (2008, 2013) shows that the effects of exogenous population changes on economic growth can be ambiguous, depending on the nature of economic progress or of the returns to specialization. However, in the latter models population growth is exogenous.

Connolly and Peretto (2003) make a step further by endogenizing fertility in the way as in the Barro and Becker (1989) model and in the presence of horizontal and vertical R&D. Under this scenario they show that in the long run, growth and fertility may diverge due to exogenous shocks (policies) affecting vertical R&D costs or horizontal innovation costs, while, again in the long run, they still move in the same direction under demographic shocks concerning the mortality rate or the cost of reproduction. However, such results are obtained through simulations, in the absence of capital accumulation and without axiomatically founded welfare criteria.

In the light of this background in the present paper we aim at taking a step further by addressing the following question: what does the relationship between endogenous economic growth, human capital and endogenous fertility look like under axiomatically founded preferences? To the best of our knowledge, this has not been done before.

In fact, while drawing on the "human capital" approach à la Lucas (1988), we depart from the existing literature as we allow for endogenous fertility[3] and put particular emphasis on the fact that, in the presence of endogenous population, welfare evaluations typically imply the comparisons between states of the world in which the size of population is different. Such a fact has two consequences: first, one need Social Welfare Orderings that are axiomatically founded also in presence of variable population. Second, one also aims at avoiding undesirable outcomes such as the so-called Repugnant Conclusion (RC henceforth; see Parfit 1976, 1984, Blackorby et al. 2002). According to the RC, any state in

contributed to such an approach and the implications of the role of population growth and public policy carry over from Jones (1995). However, Dinopoulos and Thompson (1998), Segerstrom (1998), Peretto (1998), Young (1998), and Howitt (1999) propose generalizations of Jones' (1995) model where policy affects the equilibrium growth rate.

[3] For existing works on (optimal) endogenous fertility, although without endogenous economic growth, see, for example, de la Croix et al (2012) and Pestieau and Ponthiere (2014).

which each member of the population enjoys a life above neutrality is declared inferior to a state in which each member of a larger population lives a life with lower utility (Blackorby et al. 1995, 2002). In particular, in an economic growth setting, the RC takes the form of an upper-corner solution for the population growth rate (i.e. society reproduces at its physical maximum rate, see Renström and Spataro 2011 for a discussion).

To cope with these problems we adopt a population criterion proposed in Renström and Spataro (2012), referred to as Relative Critical Level Utilitarianism (RCLU), which is in the spirit of the Critical Level Utilitarianism[4], in that it is axiomatically founded. Under RCLU the judgment (the critical level of utility for life worth living) is relative to the existing generation's level of wellbeing. In other words, according to such a criterion a society or an individual household[5] at low level of utility will set a lower threshold of utility for the next generation, and a society or an individual household with high living standard will set a higher level. So if parents had a good life, they require their children to have a good life as well, and vice versa.

To summarize, we explore the relationship between endogenous growth and population growth under RCLU, when economic growth is driven by human capital accumulation, by pinning down the conditions under which sustained long-run growth occurs and by unveiling the circumstances under which population growth and economic growth are positively or negatively correlated.

The paper is organized as follows: in section 2 we lay out the model; in section 3 we characterize the solution; in section 4 we prove local stability of the endogenous growth path and in section 5 we carry out some comparative statics. Section 6 concludes.

## 2. The model

### *2.1 Preferences*

Following Renström and Spataro (2012), we focus on a single dynasty (household) or a policymaker choosing consumption and population growth over time, so as to maximize:

[4] Critical Level Utilitarianism (see Blackorby et al. 1995), is an axiomatically founded population principle where the Critical Level is defined as a utility value (α) of an extra person, who if added to the (unaffected) population, would make society as well off as without that person.

[5] One can apply the RCLU criterion at either social level (for normative analysis) or at individual family level (for positive analysis) by aggregating over individuals that have preferences with both intergenerational altruism and reference point represented by previous generation's welfare. See next footnote.

$$W(u_{t-1}, N_t, u_{t+1}, N_{t+1}, \ldots) = \sum_{s=0}^{\infty} \quad {}^{s} N_{t+s} \left[ u_{t+s} - \sim u_{t-1+s} \right] \qquad (1)$$

where $N_t$ is the population (family) size of generation $t$, $u_t = u(c_t) \geq 0$ is the utility function of an individual of generation $t$, with $u(0) = 0$, $u' > 0, u'' < 0$, $\quad = \frac{1}{1+\circ}, \quad \in (0,1)$ the intergenerational discount factor and $\circ > 0$ the intergenerational discount rate; $\sim u_{t-1}$ is the Critical Level Utility, with $\sim \in (0,1)$ applied to generation $t$. Such a critical value is a positive function of previous generation's utility (only if $\sim u_{t-1}$ is a constant, this social ordering would coincide with CLU). Renström and Spataro (2012) refer to this population criterion as "Relative CLU" (RCLU) [6].

The continuous time version of (1) can be written as (see Appendix A):

$$U = \int_0^{\infty} e^{-\rho t} N_t u(c_t) \left[ 1 - \alpha (n_t - \rho) \right] dt \quad . \qquad (2)$$

$n_t$ is the (endogenous) population growth rate, i.e.

$$\frac{\dot{N}_t}{N_t} = n_t \qquad (3)$$

[6] In fact, if one assumes that individuals are entailed with both intergenerational altruism and relative-consumption (or relative-welfare) preferences, with reference group being previous generation's consumption (or welfare), then an individual's preferences could be written as:

$$U_t = (u_t - \sim u_{t-1}) + \quad \frac{N_{t+1}}{N_t} U_{t+1} \,,$$

such that, aggregating over individuals, we obtain:

$$W_t = N_t U_t = N_t (u_t - \sim u_{t-1}) + \quad N_{t+1} U_{t+1}$$

which coincides with eq. (1) in the text. Hence, as shown in Renström and Spataro (2012), the current analysis can be interpreted as being either normative or positive.

with $n_t \in [\underline{n}, \overline{n}]$. The integral is finite only if $(\rho - \overline{n}) > 0$, which we assume throughout the paper, implying that $1-\alpha(n_t - \rho)$>0.

*2.2. Manufacturing sector*

As in Lucas (1988) we assume Cobb-Douglas constant returns-to-scale (CRS) production technology:

$$Y_t=F(K_t,L_t)= AK_t^{¥}L_t^{1-¥} \tag{4}$$

where $A$ is the parameter representing total factor productivity, $¥ \in (0,1)$ is the output elasticity with respect to capital, $K_t$. $L_t = h_t v_t N_t$ is effective labour, $h_t$ is the human capital stock, $v_t$ is the fraction of time dedicated to work (and 1-$v_t$ the time dedicated to education), such that $v_t N_t$ is the number of individuals that are at work in each instant $t$. The capital accumulation equation is:

$$\dot{K}_t = F(K_t, L_t) - c_t N_t \tag{5}$$

Note that in eq. (5) there is no explicit cost for raising children. This is done for two reasons: first, we aim to keep or analysis in the spirit of Renström and Spataro (2011), where the same assumption is posed, and look at the very consequences of introducing RCLU in an endogenous growth model, other things being equal. Second, for the sake of tractability, we aim to be as parsimonious as possible in terms of parameters: as it will be clear in the next section, a trade-off in the choice of giving birth to an extra child clearly emerges in our model, which allows us to avoid the adoption of explicit childbearing costs of any particular form, without loss of generality[7].

*2.3 The knowledge sector*

Consider a research sector and an education sector (dissemination of knowledge). The population not working in manufacturing, $(1-v_t)N_t$, is divided between research and schooling, denoted $N_t^h$ and $N_t^s$, respectively. Each researcher discovers new knowledge, $\delta_t$, proportional to existing human capital, $h_t$, implying total research discovery is:

$$\delta_t N_t^h = \mu h_t N_t^h \tag{6}$$

where $\mu$ is a research-productivity parameter. Let $\Delta_t$ be the addition to knowledge per person through education. The entire population is educated according to a function of research discovery and educators (whose productivity is proportional to existing human capital):

$$\Delta_t N_t = \omega\left(\mu h_t N_t^h\right)^{\varepsilon}\left(h_t N_t^s\right)^{1-\varepsilon} \tag{7}$$

where $\omega$ is a schooling-productivity parameter and $\varepsilon \in (0,1)$ the elasticity of the additional knowledge with respect to research activity. Notice that the entire population is educated, including researchers and educators (to account for the fact that research and teaching is specialised, and researchers and teachers need to learn from other fields).

Efficiency in education requires allocation between researchers and educators to maximise (7), subject to $(1-v_t)N_t = N_t^h + N_t^s$. The first-order conditions (w.r.t $N_t^h$ and $N_t^s$) give

[7] For a model with CLU and taxation in the presence of childbearing costs, see Spataro and Renström (2012).

$$N_t^h = £\left(1 - v_t\right)N_t \qquad (8)$$

$$N_t^s = \left(1 - £\right)\left(1 - v_t\right)N_t \qquad (9)$$

Substituting (8) and (9) into (7) gives

$$\Delta_t N_t = {}^{-}\left(1 - v_t\right)h_t N_t \qquad (10)$$

where ${}^{-} \equiv \mu\left(«£\right)^{£}\left(1 - £\right)^{1-£}$, which is a measure of research efficiency and schooling efficiency, which in reality may differ across countries and also time.

Dividing (10) by $N_t$ we have the per-person addition to knowledge, which in continuous time is $\dot{h}_t$,

$$\dot{h}_t = {}^{-}\left(1 - v_t\right)h_t \qquad (11)$$

There is no scale effect in (11), because as the population grows larger, more individuals are needed to educate the larger population.

Since the dynasty objective (1) is also the social welfare function, and since there are no externalities, a decentralized version of the model (where firms and individuals are price takers) would yield exactly the same equilibrium (i.e. the First Welfare Theorem applies). This means we can interpret our equilibrium as either a decentralized one or as a socially optimal one (i.e. positive and normative analysis coincide). We may therefore use the terminology “socially optimal” or “individually optimal” interchangeably[8].

[8] In fact, it can be shown that the economy can be decentralized as follows. Let us suppose for simplicity that the government produces education, hiring $N_t^g = N_t^h + N_t^s$. Efficiency for the government implies (by using eq. 7):

## 3. Solution

The current-value Hamiltonian of the household's problem is the following:

$$H_t = N_t u(c_t)[1-\alpha(n_t-\rho)] + q_t[F(K_t, L_t) - c_t N_t] + \lambda_t n_t N_t + p_t(1-v_t)\text{¯}_t h_t \quad (12)$$

The term $\lambda_t n_t N_t$ in the Hamiltonian associated with eq. (3) captures the fact that at each instant of time the population size is given (and thus is a state variable) and can only be controlled by the choice of $n_t$ (which is a control variable); $q_t$ and $p_t$ are the shadow prices of physical and human capital respectively. The law of motion for the population size is provided by (3). Hence, $\lambda_t$ can be interpreted as the shadow value of population.

The first order conditions of the problem imply:

$$\frac{\partial H_t}{\partial c_t} = N_t u'_t\,[1-\sim(n_t - \text{°})] - N_t q_t = 0 \Rightarrow u'_t\,[1-\sim(n_t-\text{°})] = q_t \quad (13)$$

$$\frac{\partial H_t}{\partial n_t} = N_t(-\sim u_t + \text{ª}_t) = 0 \Rightarrow \sim u_t = \text{ª}_t \quad (14)$$

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*max* $\text{µ}(\text{«}h_t N_t^h)^{\pounds}(h_t N_t^s)^{1-\pounds} - w^g h_t(N_t^h + N_t^s)$, which gives $N_t^h = \pounds N_t^g$, $N_t^s = (1-\pounds)N_t^g$ and $\Delta_t N_t = \text{¯}h_t N_t^g$. Let us assume that the cost to the government, given by $w_t^g N_t^g$, is financed by a lump-sum tax, $T_t$. Manufacturing firms hire capital and labour services by solving: *max* $F(K_t, h_t N_t^y) - w^y h_t N_t^y - r_t K_t$. Under this scenario the household sector faces the following constraint: $\dot{K}_t = r_t K_t + w_t^y h_t v_t N_t + w_t^g h_t(1-v_t)N_t - T_t - c_t N_t$. Market clearing conditions give: $w_t^y = w_t^g = w_t$, $v_t N_t = N_t^y$, $(1-v_t)N_t = N_t^g$, $T_t = w_t h_t N_t^g$, $F_{K_t} = r_t$, $F_{L_t} = w_t$. Finally, substituting the latter conditions into the household sector budget constraint and exploiting CRS of the manufacturing production function, we get eq. (5).

$$\frac{\partial H_t}{\partial N_t} = {}^{\circ}\mathrm{a}_t - \dot{\mathrm{a}}_t \Rightarrow \dot{\mathrm{a}}_t = \left({}^{\circ} - n_t\right)\mathrm{a}_t - u_t\left[1 - \sim\left(n_t - {}^{\circ}\right)\right] - q_t\left[F_{N_t} - c_t\right]$$
(15)

$$\frac{\partial H_t}{\partial K_t} = {}^{\circ}q_t - \dot{q}_t \Rightarrow \dot{q}_t = q_t\left[{}^{\circ} - F_{K_t}\right]$$
(16)

$$\frac{\partial H_t}{\partial v_t} = q_t F_{v_t} - p_t \bar{}_t h_t = 0$$
(17)

$$\frac{\partial H_t}{\partial h_t} = q_t F_{h_t} + p_t \bar{}_t \left(1 - v_t\right) = {}^{\circ}p_t - \dot{p}_t$$
(18)

plus eqs. (3) and (5) and the transversality conditions

$$\lim_{t\to\infty} e^{-{}^{\circ}t} q_t K_t = 0, \quad \lim_{t\to\infty} e^{-{}^{\circ}t} \mathrm{a}_t N_t = 0. \quad \lim_{t\to\infty} e^{-{}^{\circ}t} p_t h_t = 0$$
(19)

In what follows we assume interiority of the solution for $n_t$, such that eq. (14) holds along the transition path.[9] By substituting for eqs. (13) and (14) into (15) we get:

$$\dot{\mathrm{a}}_t = -u_t - u'_t\left[1 - \sim\left(n_t - {}^{\circ}\right)\right]\left[F_{N_t} - c_t\right].$$
(20)

We obtain four dynamic equations that, together with the transversality conditions, fully characterize our dynamic system (from now on we omit time subscripts for the sake of

[9] For corner $n_t$ the economy would behave as in Lucas (1988).

notation):

$$\frac{\dot{c}}{c} = -\frac{u}{\alpha u' c} + \frac{G}{\alpha}\left[1 - \frac{F_N}{c}\right]$$
(21)

$$\frac{\dot{k}}{k} = \frac{F_K}{¥} - \frac{c}{k} - n$$
(22)

$$\frac{\dot{h}}{h} = ^{-}(1 - v)$$
(23)

$$\dot{n} = \frac{G}{\alpha}\left[\frac{u''}{u'}\dot{c} - \left(\rho - F_K\right)\right] = G\left\{\frac{u''}{u'}\left[-\frac{u}{\alpha u'} + \frac{G}{\alpha}\left(c - F_N\right)\right] - \left(\rho - F_K\right)\right\}$$
(24)

where $G \equiv 1 + \alpha(\rho - n) > 0$. Eq. (21) is obtained by taking the time derivative of eq. (14) and combining with eq. (20), eq. (22) is (5) in per capita terms, where $k \equiv K / N$, (23) stems from eq. (11) and (24) stems from the time derivative of (13) and combining with eq. (16), (13) and (21)**.**

We now briefly comment on aspects of the solution. In particular eq. (22) states that, along the transition path, both the of growth of population, *n*, and consumption, *c*, must satisfy the resources available for the economy, while. eq. (23) recovers the law of accumulation of human capital. Moreover, eq. (21) states that at the optimum both consumption and fertility should be chosen in such a way the rate of growth of consumption is proportional to the difference between the increase of social welfare due to an extra individual at the margin, *u*, and the marginal value (in utility units) of what a newborn takes out of society, $Gu'[c-F_N]$, which is positive due to the presence of a positive capital stock.

In order to simplifying the analysis of the dynamic system, let us define:

$$\tilde{h} \equiv \frac{vh}{k};$$

(25)

From (17) and (18) it follows:

$$\frac{\dot{p}}{p} = \circ + {}^{-}(1-v) - {}^{-}h\frac{F_h}{F_v} = \circ - {}^{-}$$

(26)

where we have exploited $\frac{F_h}{F_v} = \frac{v}{h}$. Moreover, substituting for $F_v = F_L hN$ into (17) and exploiting $F_L = (1-\text{¥})A\tilde{h}^{-\text{¥}}$, time derivative of (17) is:

$$\frac{\dot{q}}{q} + \frac{\dot{N}}{N} - \text{¥}\frac{\dot{\tilde{h}}}{\tilde{h}} = \frac{\dot{p}}{p}$$

(27)

such that, by exploiting (3), (16), and (26), we have:

$$\frac{\dot{\tilde{h}}}{\tilde{h}} = \frac{{}^{-} + n - F_K}{\text{¥}}.$$

(28)

Note that eq. (28) expresses growth in capital and human capital as a function of population growth rate and marginal productivity of capital, which however are endogenous (to be explored below).

*3.1. Balanced growth path (BGP)*

Along the balanced-growth path $\dot{\tilde{h}}$ =0, such that, from (28):

$$F_K = {}^{-} + n \tag{29}$$

Moreover, by equating (22) and (23) and using (29) we get

$$\frac{c}{k} = v^{-} + \frac{1-¥}{¥}\left({}^{-} + n\right) \tag{30}$$

Finally, from eq. (24), a BGP, where $\dot{n} = 0$, implies

$$\frac{\dot{c}}{c} = \frac{F_K - \rho}{\pm(c)} \tag{31}$$

where $\pm\left(c\right) = -\frac{u''c}{u'} > 0$ is the inverse of the intertemporal elasticity of substitution (IES) of consumption. Throughout the paper we will assume the constancy of such a IES, by adopting a CES function for utility.

Moreover, by equating (31) and (23) it follows:

$$F_K = \pm^{-}(1-v) + \rho \tag{32}$$

and from (29) and (32):

$$n = \pm^{¯}(1 - v) + \rho -^{¯} .$$

(33)

At first sight eq. (33), combined with eq. (11), indicates a positive relationship between population growth and economic growth. However, such a relationship is more complex, given that *v* enters both equations and is, in turn, endogenous. After some algebra (see Appendix B for details), it can be shown that the BGP value for *v* is the solution of the following second order equation for *v*:

$$(¯v)^2 + \frac{(ab + m)¯}{1 - a} v - \frac{bm}{1 - a} = 0$$

(34)

with $a \equiv \frac{1 - 2¥}{¥} \pm$ , $b \equiv ¯ + \frac{1}{\sim} \frac{1}{1 - \pm}$ , $m \equiv \frac{1 - ¥}{¥} \left( ° + \pm ¯ \right)$ .

For $a$<1, the (strictly) positive root of (34) is:

$$v_1 = -\frac{1}{2¯} \frac{ab + m}{1 - a} + \frac{1}{¯} \sqrt{\frac{1}{4} \frac{(ab + m)^2}{(1 - a)^2} + \frac{bm}{1 - a}} .$$

(35)

and for $a \geq 1$

$$v_2 = \frac{1}{2¯} \frac{ab + m}{a - 1} - \frac{1}{¯} \sqrt{\frac{1}{4} \frac{(ab + m)^2}{(1 - a)^2} + \frac{bm}{1 - a}} .$$

(36)

The intuition of the economic rationale behind eqs. (35) and (36) will be provided in section 5, where a comparative statics analysis is carried out.

Finally, plugging such solutions for *v* into (32), (33), (30) and (23), gives the balanced-growth values for $F_K$, *n*, *c/k* and the growth rate of the economy, respectively.[10]

Note that, in order to have interior solution for *v*, it must be that *v*<1. This implies, both for $v_1$ and $v_2$ that the following inequality, stemming from (34), must be satisfied:

$$(1-a)¯^{2}+(ab+m)¯-bm\geq 0 \qquad (37)$$

The latter condition identifies a set of restrictions on the parameters that insures interiority of the solution for *v*.

More precisely, we have:

**Proposition 1:** *Necessary and sufficient for having v*<1 *is*:

$$¯\geq ¯^{*}=\frac{\pm}{2\alpha(1-\pm)}+\frac{1}{2\alpha}\sqrt{\left(\frac{\pm}{1-\pm}\right)^{2}+4\alpha°\,\frac{1-¥}{1-\pm}} \qquad (38)$$

**Proof:** By exploiting the definitions of *a*, *b* and *m*, $¯^{*}$ is the positive root to (37).[11] □

[10] The transversality conditions (19) hold under the endogenous growth path. To see this, let *g* denote the balanced growth rate, then (using (16) and (3)) we have $\frac{d}{dt}(qK)=(°-F_K+n+g)qK$, which when integrated, together with (19) gives $e^{-°t}q_tK_t=q_0K_0e^{-(F_K-n-g)t}=q_0K_0e^{-¯vt}$ (where the last equality follows from (23) and (29)). Next, by (14), we have *λN=αuN*, thus $\frac{d}{dt}(ªN)=\left(\frac{u'}{u}\frac{\dot{c}}{c}+n\right)ªN=[(1-\pm)g+n]ªN$, where the last equality follows from the iso-elastic utility function. Integrating, together with (19), gives $e^{-°t}ª_tN_t=ª_0N_0e^{-[°-n-(1-\pm)g]t}=ª_0N_0e^{-¯vt}$ (where the last equality follows from (23) and (33)). Finally, using (26), we have $\frac{d}{dt}(ph)=(°-¯+g)ph$, which when integrated, together with (19), gives $e^{-°t}p_th_t=p_0h_0e^{-(¯-g)t}=p_0h_0e^{-¯vt}$ (where the last equality follows from (23)). Thus all three terms in (19) go to zero as t→∞.

[11] We have written inequality (38) as a restriction of *ϑ*, as a function of the other parameters. Of course we could have rewritten the inequality to provide restrictions on another parameter, as a function of *ϑ*.

Proposition 1 states that some economies can be trapped at zero (per-capita) growth when educational efficiency is too low ( $\bar{\ }$ too low): in that case society (or households) finds it convenient to employ all labour force (or work) in the manufacturing sector (with $v$=1 and, thus, $\frac{\dot{h}}{h} = \bar{\ }(1-v) = 0$ ).

*3.2. Zero growth steady state*

If $v$ is at its upper corner ($v$=1), i.e. (38) is violated, balanced growth is not achievable. In this case

$$\frac{\dot{c}}{c} = \frac{\dot{h}}{h} = \bar{\ }(1-v) = 0 \quad . \tag{39}$$

Hence, by (31) (or 28)

$$F_K = \circ \tag{40}$$

which univocally pins down the steady-state capital intensity. Moreover, eq. (17) now becomes:

$$\frac{\partial H}{\partial v_t} = q_t F_{v_t} - p_t \bar{\ }_t h_t > 0 \tag{41}$$

In this case the steady state values of $c$ and $\circ - n$ are provided by the system of equations (21) and (22), both being equal to zero, which yield:

$$\rho - n = \frac{c - F_N}{k} = \frac{\pm}{2\alpha(1-\pm)} + \frac{1}{2\alpha}\sqrt{\left(\frac{\pm}{1-\pm}\right)^2 + 4\frac{\alpha\circ(1-\yen)}{(1-\pm)}} \qquad (42)$$

Note that since (38) is violated, $\bar{\ }^* > \bar{\ }$, and eq. (29) is now an inequality:

$$\circ - n > \bar{\ }. \qquad (43)$$

To summarize the analysis carried out so far, we can state that in the present model a crucial condition for sustained long-run growth to emerge is that the efficiency of human capital production, represented by $\bar{\ }$, is sufficiently high. In fact, if returns to human capital investment are too low, income per capita will be constant and the economy will be entrapped in a zero per-capita-growth regime, where aggregate quantities are driven by population growth. On the other hand, if educational efficiency is large enough, then society (or household) will find it convenient to invest resources in the education sector, so that a BGP regime will emerge.

In fact, in the BGP regime the relationship between the economic growth and population growth is nontrivial, and will be analyzed in the section that follows.

Finally, it can be shown that necessary for avoiding the RC is that $\sim > 0$ (i.e. the proof provided in Renström and Spataro (2012) applies also our model with human capital accumulation).

## 4. Stability

We now analyze the local stability of the BGP equilibrium[12]. First, let us define $\tilde{c} \equiv \frac{c}{k}$, which is constant along the BGP. We can reformulate the dynamic equations characterizing our economy as follows:

$$\frac{\dot{\tilde{c}}}{\tilde{c}} \equiv -\frac{1}{\alpha(1-\pm)} + \frac{G}{\alpha}\left(1-\frac{F_N}{k\tilde{c}}\right) - \frac{F_K}{\yen} + \tilde{c} + n = \frac{(1+\sim^{\circ})(1-\pm)-1}{\alpha(1-\pm)} - \frac{F_K}{\yen}\left[\frac{G}{\alpha\tilde{c}}(1-\yen)+1\right] + \tilde{c} \quad (44)$$

$$\frac{\dot{\tilde{h}}}{\tilde{h}} = \frac{\bar{\;} + n - F_K}{\yen} \quad (45)$$

$$\dot{n} = -\frac{G}{\alpha}\left[\frac{-\pm + \rho\alpha(1-\pm)}{\alpha(1-\pm)} - F_K\left[\frac{(1-\yen)}{\yen}\frac{G\pm}{\sim\tilde{c}} + 1\right] + \pm\frac{G}{\sim}\right] \quad (46)$$

The associated Jacobian matrix is:

$$J = \begin{bmatrix} \mu + \tilde{c} & -\tilde{c}\,\mu\,\frac{(1-\yen)G + \tilde{c}\sim}{\tilde{h}G} & \frac{\tilde{c}\,\mu\sim}{G} \\ 0 & -\frac{\tilde{c}\,\mu\sim}{G} & \frac{\tilde{h}}{\yen} \\ -\frac{\mu G\pm}{\sim\tilde{c}} & \mu\,\frac{G\pm(1-\yen)+\yen\sim\tilde{c}}{\sim\tilde{h}} & \pm\,\frac{G-\mu\sim}{\sim} \end{bmatrix}$$

[12] As for the local stability of the zero-growth equilibrium, it has been analyzed in Renström and Spataro (2012).

where $\mu \equiv \frac{(1-¥)}{¥}\frac{F_K G}{\sim\tilde{c}} > 0$, $\mu < \frac{G}{\sim}$, $F_{K\tilde{h}} = (1-¥)\frac{F_K}{\tilde{h}}$.

Hence, we can provide the following Proposition:

**Proposition 2:** *The balanced growth path is locally stable.*

**Proof:** See Appendix C.

□

## 5. Comparative statics

In this section we carry out some comparative statics analyses in order to characterize the role of the models' parameters in affecting the economic growth rate and the population growth rate.

### *5.1. The role of deep parameters of the model*

The results of the analysis can be summarized through the following Proposition:

**Proposition 3:** *The balanced growth-rate is increasing in* ¯, ¥ ~ *and decreasing in* °, ± . *The rate of growth of population is increasing in* ¥, ~ *and decreasing in* ¯, ° *and ambiguous in* ± .

**Proof:** See Appendix D.

□

We summarize Proposition 3 in Table 1:

**Table 1. The effect of parameters on equilibrium growth**

| Parameters/Variables | *v* | Balanced growth rate | *n* |
|---|---|---|---|
| ¯ | - | + | - |
| ¥ | - | + | + |
| ~ | - | + | + |
| ° | + | - | + |
| ± | + | - | +/- |

Some comments on the findings of the comparative statics exercises that we have carried out are worth doing.

First, according to our model, differences in long-run growth rates among countries may depend on differences in both 1) preferences and 2) technology, the latter being concerned with both i) the manufacturing sector and ii) the education sector.

1) As for preferences, an increase in the critical level parameter, ~ , will make consumption more costly for society (in that future generations wellbeing will be more demanding in terms of resources required to current generations). This will induce individuals to devote more resources to the accumulation of both physical and human capital. As a consequence, economic growth will increase. However, since wellbeing of future generations will increase relative to current generation's, society finds room to increase total welfare by bringing to life extra-individuals.

On the other hand, an increase of the discount rate, ° , by making wellbeing of future generations less relevant, will make it less costly for society to reduce future consumption. This will be possible by increasing the number of children per household on the one hand (increase in *n;* recall that utility function is linear in the population size) and by reducing the pace of accumulation of both human and physical capital on the other hand (i.e. reducing future per-capita consumption; recall that social welfare is concave in per-capita consumption).

2) As for technology:

i) in the manufacturing sector, an increase of ¥ will cause an increase in the factor price ratio between capital and labour, such that individuals will tend to move from the manufacturing sector to the education sector. This will imply higher accumulation of human capital and higher per-capita growth, which leaves room for more individuals to be brought into life.

ii) as for the education sector, as already stressed in the previous section, the parameter measuring the efficiency of the production function of human capital, $\bar{\ }$, is capable to produce a qualitative switch in the growth regime. In fact, under the zero-growth rate regime ($\bar{\ } < \bar{\ }^*$), changes in the latter parameter do not produce any effect on the economy's equilibrium. However, increases of $\bar{\ }$ beyond such a threshold, by making investment in human capital more attractive, will produce an increase of the latter and thus, of per-capita income growth rate. Since such a shift will generate a less than proportional increase in the steady state value of marginal productivity of capital (see eq. 28), according to eq. (29) the population growth rate must necessarily decrease.

*5.2. The relationship between balanced growth and population dynamics*

In this section we illustrate our results, by comparing them with the ones delivered by the existing literature.

Recall that in most previous work (Uzawa-Lucas and the semi-endogenous growth models a la Jones 1995), fertility affects positively the economic growth and, moreover, is the only engine of economic growth. However, our model tells a somehow different story. In fact, by eqs. (23) and (33) we get that the economic growth rate associated with the BGP is:

$$g \equiv \bar{\ }(1-v) = \frac{n}{\pm} + \frac{\bar{\ } - \rho}{\pm} \tag{56}$$

which is positive even if population growth is null. In fact, in our model the engine of growth is the accumulation of human capital, and, in particular, by the effectiveness of such a process, whose returns are measured by $\bar{\ }$.

Furthermore, according to the above expression, it is still possible that higher fertility is associated with higher growth. The most straightforward example is the case of higher critical level $\tilde{\ }$ that positively affects *g* only through *n*.

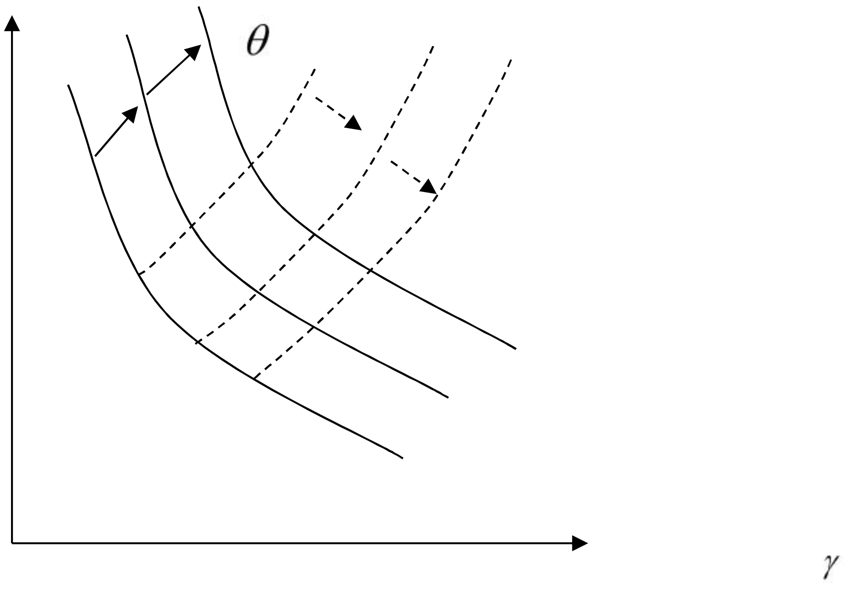
However, we have also shown that lower population growth does not necessarily imply lower economic growth, in that *n* is also a function of parameters that affect *g*. Similarly, according to our results higher population growth rates can be associated with lower economic growth rates.

To make our point clearer, we depict our results in Figure 1.

The solid lines are "iso-g" lines, that is, the loci of all combinations of the parameters (in this case $^{-},\yen$ ) that provide the same economic growth rate *g*. According to Proposition 3, such locus is negatively sloped and the associated *g* increases pointing north-east. The dotted lines are the "iso-n" lines, that is the loci of all combinations of parameters that entail the same rate of growth of population. Our results imply that these loci are positively sloped and the associated *n* increases going south-east. Take for example two points in this Figure (B and C): in the first case low population growth is associated with high levels of economic growth, while in the latter a high level of population growth is associated with low economic growth.

We can conclude that in our paper, the link between population dynamics and economic development is weakened, on one hand, because the former is no longer essential for the latter, and enriched, on the other hand, since more combinations are possible between the two variables, depending on the fundamental parameters of the economy. Thus we argue that the different combinations of such fundamental parameters might be at the origin of the observed cross-countries differences in long-run performances. The analysis of this argument is left for future research.

**Figure 1: Iso-growth curves as functions of parameters**



## 6. Conclusions

In this work we have analyzed the long run relationship between population growth, human capital accumulation and economic growth. For doing this we have adopted a framework that entails both endogenous economic growth and endogenous fertility. Moreover, we have assumed a Social Welfare function that is axiomatically founded, purely Welfarist and that allows to avoid the Repugnant Conclusion (that is upper-corner solutions for population growth).

Under these assumptions we have shown that the take-off regime of sustained long-run economic growth can only take place when the efficiency of human capital accumulation (the education sector) reaches a certain threshold. Below such a level, increases of the efficiency in the education system produce no effect on the economy, which will continue to stick at its zero per-capita-growth regime. On the other hand, beyond such a threshold sustained growth does occur, and further increases in efficiency in the education sector will generate an increase of the economic growth rate, an increase in human capital accumulation, and a decrease of population growth. The latter results seem in line with the empirical findings concerning the co-movements of the above mentioned variables in the last decades (see, for example, Galor 2005).

Finally, we have shown that in the long run population growth and economic growth can diverge, in that positive and high economic growth rates can be associated with low levels of population growth, and vice-versa. Since both variables depend on the parameters of the underlying economy, the exact shape of their relationship is ultimately an empirical matter.

As for policy implications, according to our model any policy aimed at producing the conditions for underdeveloped countries to escape from poverty traps and to enter the regime of sustained growth should be focused on the enhancement of human capital accumulation, that is on the development of the education sector of such countries. The analysis of the effects of public expenditure on the education sector is left for future research.

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## Appendix A: The form of eq. (2) (drawn from Renström and Spataro 2012).

By starting from eq. (1) and collecting utility terms of the same date, the welfare function *W* can be written as:

$$W = \sum_{t=0}^{\infty} {}^{t} N_t u(c_t)[1 - \sim \ (1+n_t)] - \sim N_0 u(c_{-1})$$

(A.1)

By ignoring $c_{-1}$ as it is irrelevant for the planning horizon, and defining $= \frac{1}{1+\circ}$ we get:

$\sum_{t=0}^{\infty}\left(\frac{1}{1+\circ}\right)^{t} N_t u(c_t)\left(1 - \sim \frac{1+n_t}{1+\circ}\right)$. In continuous time, by approximating $\frac{1+n_t}{1+\circ} \approx -(\circ - n_t)$ the latter expression can be written as follows:

$$U = \int_0^{\infty} e^{-\circ t} N_t u(c_t)[1 - \sim (n_t - \circ)]dt \qquad \square$$

(A.2)

## Appendix B: The BGP value of *v*

By equating eq. (21) to (23) we get:

$${}^{-}(1-v) = -\frac{1}{\alpha(1-\pm)} + \frac{[1+\sim(\circ - n)]}{\alpha}\left[1 - \frac{F_N}{c}\right].$$

(B.1)

Equation (B.1), using (29) and recalling that $\frac{F_N}{c} = \frac{1-¥}{¥}\frac{k}{c}F_K$, becomes:

$$^{-}v = ^{-} + \frac{1}{\alpha(1-\pm)} - \frac{\left[1+\sim\left(-\pm^{-}(1-v)+^{-}\right)\right]}{\alpha}\left[1-\frac{1-¥}{¥}\frac{k}{c}\left(\pm^{-}(1-v)+\rho\right)\right]$$
(B.2)

Finally, by using (30) and (33), eq. (B.2) can be written as:

$$^{-} - ^{-}v + \frac{1}{\alpha(1-\pm)} = (v^{-})\frac{\frac{1}{\alpha}+^{-}(1-\pm)+\pm^{-}v}{v^{-}\left(1-\frac{1-¥}{¥}\pm\right)+\frac{(1-¥)}{¥}\left(\pm^{-}+\circ\right)},$$
(B.3)

which provides the following second order equation for *v*:

$$(^{-}v)^{2} + \frac{(ab+m)^{-}}{1-a}v - \frac{bm}{1-a} = 0$$
(B.4)

with $a \equiv \frac{1-2¥}{¥}\pm$, $b \equiv ^{-} + \frac{1}{\sim}\frac{1}{1-\pm}$, $m \equiv \frac{1-¥}{¥}\left(\circ + \pm^{-}\right)$.

For $a<1$, the (strictly) positive root of (B.4) is:

$$v_1 = -\frac{1}{2^{-}}\frac{ab+m}{1-a} + \frac{1}{^{-}}\sqrt{\frac{1}{4}\frac{(ab+m)^{2}}{(1-a)^{2}}+\frac{bm}{1-a}}.$$
(B.5)

For $a>1$, eq. (B.4) has two positive roots; however, since the argmax ($\hat{v}$) of the parabola in eq. (B.4) in *v* is bigger than 1, i.e.

$$\hat{v} = -\frac{1}{2^{-}}\frac{ab+m}{1-a} = \frac{\frac{1-2¥}{¥}\pm\left(^{-}+\frac{1}{\sim}\frac{1}{1-\pm}\right)+\frac{1-¥}{¥}\left(° + \pm^{-}\right)}{2^{-}\left(\frac{1-2¥}{¥}\pm -1\right)} \geq 1 \Leftrightarrow {}^{-}\left(2+\pm\right)+\frac{1-¥}{¥}° + \frac{1-2¥}{¥}\frac{\pm}{\sim}\frac{1}{1-\pm} \geq 0$$

then the only (strictly positive) root of eq. (B.4) that can be lower than 1 is

$$v_2 = \frac{1}{2^{-}}\frac{ab+m}{a-1} - \frac{1}{^{-}}\sqrt{\frac{1}{4}\frac{(ab+m)^2}{(1-a)^2}+\frac{bm}{1-a}}\ .$$

(B.6)

**Appendix C: Proof of Proposition 2**

Let ¤ be the eigenvalues, then the characteristic equation associated with the Jacobian matrix above is:

$$-¤^3 + ¤^2\Omega_1 + ¤\Omega_2 + \Omega_3 \equiv \Pi(¤) = 0$$

(C.1)

where

$$\Omega_1 = tr(J) = \frac{1}{G\sim}\left\{\sim\tilde{c}\left(G-\sim µ\right)+G\sim µ\left(1-\pm\right)+\pm G^2\right\} > 0$$

(C.2)

$$\Omega_2 = \frac{\sim µ\tilde{c}}{G}\left[(1-\pm)µ+\tilde{c}\right]+2\tilde{c}µ\left(\frac{1}{2}+\pm\right)+\frac{G\pm}{\sim¥}\left[µ(1-¥)-\tilde{c}\right]$$

(C.3)

$$\Omega_3 = \det(J) = -\frac{\tilde{c}\,µ}{G{\sim}¥}\left\{{\sim}Gµ\left[(1+±)¥ - ±\right] + ±G^2(1-¥) + \tilde{c}¥{\sim}\left[G + ±\left(G - {\sim}µ\right)\right]\right\}$$

(C.4)

Since we do not obtain closed solutions to eq. (C.1), we characterize the shape of $\Pi(¤)$.

First of all, $\lim_{¤\to\pm\infty}\Pi(¤) = \mp\infty$ and $\Pi(0) = \Omega_3$. Moreover, first derivative:

$$\Pi'(¤) = -3¤^2 + 2\Omega_1¤ + \Omega_2$$

(C.5)

has a positive argmax in $¤$ and $\lim_{¤\to\pm\infty}\Pi(¤)' = -\infty$. Hence, $\Pi(¤)$ is either always decreasing (if (C.5) has no real roots then $\Pi'(¤)$ is always negative) or it is increasing in the interval, $(¤_1, ¤_2)$, (which are the smaller and the larger roots to (C.5) respectively). In either cases, we recall that stability is ensured by the existence of one negative root to eq. (C.1). The analysis carried out so far implies that sufficient for eq. (C.1) to have only one negative root is that $\Omega_3$ <0. We now show that $\Omega_3$ <0.

By exploiting the definition of $µ$ and rearranging terms, (C.4) becomes:

$$\Omega_3 = -(1+±)µ\left[F_K\frac{G}{\sim}² + ¢\left(\frac{G}{\sim} - F_K\right)\tilde{c} + \tilde{c}^2\right]$$

(C.6)

with $² \equiv \left(\frac{1-¥}{¥}\right)\left(1 - \frac{1}{¥}\frac{±}{1+±}\right)$ and $¢ \equiv \left(\frac{1-¥}{¥}\right)\left(\frac{±}{1+±}\right)$. Next, substituting for $\tilde{c}^2$ in (C.6) from the identity $\tilde{c}^2 \equiv \left(\tilde{c} - \frac{1-¥}{¥}F_K\right)^2 + 2\frac{1-¥}{¥}F_K\left(\tilde{c} - \frac{1-¥}{¥}F_K\right) - \left(\frac{1-¥}{¥}F_K\right)^2$, where $\left(\tilde{c} - \frac{1-¥}{¥}F_K\right) = F_N > 0$ and rearranging terms, we get:

$$\Omega_3 = -(1+\text{±})\left\{\text{µ}\left(\frac{1-\text{¥}}{\text{¥}}\right)F_K\left[\frac{G}{\sim}\text{²} + \text{¢}\left(\frac{G}{\sim} - F_K\right) + \left(\frac{1-\text{¥}}{\text{¥}}\right)F_K\right] + \right.$$
$$\left. + \left[\text{¢}\left(\frac{G}{\sim} - F_K\right) + 2\left(\frac{1-\text{¥}}{\text{¥}}\right)F_K\right]\left(\tilde{c} - \frac{1-\text{¥}}{\text{¥}}F_K\right) + \left(\tilde{c} - \frac{1-\text{¥}}{\text{¥}}F_K\right)^2\right\}.$$

Since $\text{²} + \text{¢} > 0$ and $\left(\frac{1-\text{¥}}{\text{¥}}\right) - \text{¢} > 0$, the first expression in square brackets is positive. Moreover, since $2\left(\frac{1-\text{¥}}{\text{¥}}\right) - \text{¢}$ >0, also the expression in the second square brackets is positive. Hence, we can conclude that $\Omega_3$ <0

□

**Appendix D: Proof of Proposition 3**

Preliminarily, let us rewrite eq. (34) as follows:

$$(1-a)(\bar{v})^2 + (ab+m)\bar{v} - bm = 0$$
(D.1)

and differentiate it with respect to the parameters, such that:

$$[2(1-a)(\bar{v}) + ab + m]d(\bar{v}) + \bar{v}(b - \bar{v})da + (a\bar{v} - m)db + (\bar{v} - b)dm = 0$$
(D.2)

by recognizing from (D.1) that $ab + m = \frac{bd - (1-a)(\bar{v})^2}{\bar{v}}$ and plugging it into the first term in square brackets of (D.2) we get that the latter term is:

$$\Delta \equiv \left[(1-a)(\text{¯}v)+\frac{bm}{\text{¯}v}\right]=\frac{1}{\text{¯}v}\left[m(b-\text{¯}v)+b(m-a\text{¯}v)\right]$$
(D.3)

Given that $(b-\text{¯}v)=\text{¯}(1-v)+\frac{1}{\sim}\frac{1}{1-\pm}>0$
(D.4)

and $(a\text{¯}v-m)=-\left\{\left(\frac{1-\text{¥}}{\text{¥}}\right)\left[\circ+\pm\text{¯}(1-v)\right]+\pm\text{¯}v\right\}<0$
(D.5)

It also follows that $\Delta$ >0.

**D.1. The effect of ¯ .**

When ¯ varies, we get that:

$da=0$, $db=d\text{¯}$, $dm=\left(\frac{1-\text{¥}}{\text{¥}}\right)\pm d\text{¯}$ .

Preliminarily, note that, from eq. (D.2):

$$\frac{d(\text{¯}v)}{d\text{¯}}=\frac{m-a\text{¯}v+\left(\frac{1-\text{¥}}{\text{¥}}\right)\pm(b-\text{¯}v)}{\Delta}$$

(D.6)

which, by eqs. (D.3), (D.4) and (D.5) is positive.

Next, we can write:

$$-\frac{dv}{d¯}=\frac{d\left(¯v\right)}{d¯}-v=\frac{m-a¯v+\left(\frac{1-¥}{¥}\right)±\left(b-¯v\right)}{\Delta}-v,$$

which yields:

$$-\frac{dv}{d¯}=\frac{1}{\Delta}\left\{\left(\frac{1-¥}{¥}\right)\left[°\left(1-2\frac{b}{¯}\right)+\left(¯-b\right)±\right]+\left[a\left(¯-b\right)+\left(\frac{1-¥}{¥}\right)°\right]v\right\}$$

(D.7)

Note that, while the sign of the first term in square brackets is negative, the one of the second term in square brackets can be ambiguous. If it is negative, then $\frac{dv}{d¯}$ >0; if it is positive, then the sign of (D.7) is ambiguous; in the latter case we can check whether, for $v$=1, the whole expression in (D.7) can be positive as well. In fact, we get that, for $v$=1, (D.7) becomes:

$$-\frac{dv}{d¯}=\left(\frac{¯-b}{¯}\right)\frac{2\left(\frac{1-¥}{¥}\right)°+¯±}{\Delta}<0.$$

(D.8)

Hence, we can conclude that $\frac{dv}{d¯}$ <0.

As for the economy's rate of growth $\frac{\dot{h}}{h} = ¯(1-v)$, we get that

$$\frac{d\left(\frac{\dot{h}}{h}\right)}{d¯} = (1-v) - ¯\frac{dv}{d¯} > 0$$
(D.9)

Finally, as for $n = ¯(\pm -1) - \pm ¯ v + \circ$, one gets:

$$\frac{dn}{d¯} = (\pm -1) - \pm v - \pm \frac{d(¯v)}{d¯} < 0$$
(D.10)

by eq. (D.6).

**D.2. The effect of ¥ .**

When ¥ varies, we get that:

$da = -\frac{\pm}{¥^2} d¥$, $db = 0$, $dm = -\frac{\circ + ¯\pm}{¥^2} d¥$. Substituting in to (B.2) and collecting terms, we get:

$$-\frac{dv}{d¥} = -\frac{1}{\Delta}\frac{(b - ¯v)}{¥^2}\left[\circ + \pm ¯(1-v)\right] < 0$$
(D.11)

As for the economy's rate of growth $\frac{\dot{h}}{h} = ¯(1-v)$, we get that

$$\frac{d\left(\frac{\dot{h}}{h}\right)}{d\text{¥}} = -\text{¯}\frac{dv}{d\text{¥}} > 0$$
(D.12)

Finally, as for $n = \text{¯}(\pm -1) - \pm\text{¯}v + \text{°}$ , one gets:

$$\frac{dn}{d\text{¥}} = -\text{¯}\pm\frac{dv}{d\text{¥}} > 0$$
.
(D.13)

**D.3. The effect of ~ .**

When ~ varies, we get that:

$da = 0$, $db = -\frac{1}{\sim^2}\left(\frac{1}{1-\pm}\right)d\sim$ , $dm = 0$. Substituting in to (B.2) and collecting terms, we get:

$$[2(1-a)(\text{¯}v) + ab + m]d(\text{¯}v) + (a\text{¯}v - m)db = 0$$
(D.14)

such that,

$$\text{¯}\frac{dv}{d\sim} = \frac{1}{\Delta}(a\text{¯}v - m)\frac{1}{\sim^2}\left(\frac{1}{1-\pm}\right) < 0$$
(D.15)

As for the economy's rate of growth $\frac{\dot{h}}{h} = {}^{-}(1-v)$, we get that

$$\frac{d\left(\frac{\dot{h}}{h}\right)}{d\sim} = -{}^{-}\frac{dv}{d\sim} > 0$$
(D.16)

Finally, as for $n = {}^{-}(\pm -1) - \pm{}^{-}v + \circ$, one gets:

$$\frac{dn}{d\sim} = -\pm{}^{-}\frac{dv}{d\sim} > 0 \; .$$
(D.17)

**D.4. The effect of $\circ$ .**

As for the effect of $\circ$ **,** when the latter parameter changes one obtains:

$$da = 0, \; db = 0, \; dm = \left(\frac{1-\yen}{\yen}\right)d\circ \; .$$

Substituting into (D.2) it descends:

$$-\frac{dv}{d\circ} = -\frac{1}{\Delta}\left({}^{-}v - b\right)\left(\frac{1-\yen}{\yen}\right) > 0$$
(D.18)

As for the economy's rate of growth $\frac{\dot{h}}{h} = ¯(1-v)$, we get that

$$\frac{d\left(\frac{\dot{h}}{h}\right)}{d°} = -¯\frac{dv}{d°} < 0$$

(D.19)

Finally, as for $n = ¯(± - 1) - ±¯v + °$, one gets:

$$\frac{dn}{d°} = -¯± \frac{dv}{d°} + 1 .$$

(D.20)

which, in principle, could take any sign. However by expanding it we get:

$$\frac{dn}{d°} = \frac{1}{\Delta}\left[2bm - (ab+m)¯v - ±(b - ¯v)\left(\frac{1-¥}{¥}\right)¯v\right] .$$

(D.21)

The source of ambiguity of the sign of (D.21) are the second and the third term in the square brackets. However, by checking whether, for *v*=1, (D.21) is positive or negative, we get:

$$\frac{dn}{d°} = \frac{1}{\Delta}\left\{\left[b\left(\frac{1-¥}{¥}\right)° + ±¯\right] + \left(\frac{1-¥}{¥}\right)°(b - ¯)\right\} > 0 .$$

(D.22)

Hence, we can conclude that $\frac{dn}{d°}$ >0 for any level of *v*.

**D.5. The effect of $\pm$ .**

When $\pm$ varies, we get that:

$da = \frac{1-2¥}{¥}d\pm$ , $db = \frac{1}{\sim}\left(\frac{1}{1-\pm}\right)^{2}d\pm$ , $dm = ¯\left(\frac{1-¥}{¥}\right)d\pm$ . Substituting in to (D.2) and collecting terms, we get:

$$-\frac{dv}{d\pm} = \frac{1}{\Delta}\left\{(b-¯v)\frac{¯}{¥}\left[(1-¥)(1-v)+¥v\right]+(m-a¯v)\frac{1}{\sim}\left(\frac{1}{1-\pm}\right)^{2}\right\} > 0$$
(D.23)

As for the economy's rate of growth $\frac{\dot{h}}{h} = ¯(1-v)$ , we get that

$$\frac{d\left(\frac{\dot{h}}{h}\right)}{d\pm} = -¯\frac{dv}{d\pm} < 0$$
(D.24)

Finally, as for $n = ¯(\pm - 1) - \pm¯v + °$ , one gets:

$$\frac{dn}{d\pm} = ¯(1-v) - ¯\pm\frac{dv}{d\pm}$$
(D.25)

which is ambiguous. In fact, we can evaluate such a derivative at the extremes of the existence interval. Such an interval is defined by $\pm = 0$ and $\pm^{*}$ (where $v$=1):

$$\left.\frac{dn}{d\pm}\right|_{\pm=0} = {}^{-}(1-v|_{\pm=0}) > 0$$

where $v|_{\pm=0} = -\frac{m}{2^{-}} + \frac{1}{-}\sqrt{\frac{m^2}{4} + bm}$

and $\left.\frac{dn}{d\pm}\right|_{\pm=\pm*} = -\pm^{-} \left.\frac{dv}{d\pm}\right|_{v=1} < 0$

Hence, we can conclude that the derivative of *n* with respect to $\pm$ can change sign.