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Abstract

This paper presents sufficient conditions for existence and uniqueness of a steady state equilibrium in an OLG model with non-separable preferences and analyzes the implications of such assumption for the local stability of the steady state equilibrium. The conditions for a stable solution are derived under the assumption that habits are transmitted *both* across and within generations. Under this assumption, monotonic convergence to the steady state is not always assured. Both competitive and optimal equilibrium may display explosive dynamics.

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1 Introduction

This paper derives sufficient conditions for existence and uniqueness of a steady state equilibrium in an OLG model with non-separable preferences and analyzes the implications of non-separable preferences for the local stability of the steady state equilibrium.

The main assumption of this paper is that habits are transmitted *both* across and within generations, i.e. habits are transmitted from one generation to the next one (intergenerational spillover) *and* from one period to the next one (intragenerational spillover). This assumption is modeled in the paper with non-separable preferences *both* across and within generations.

In the literature on stability of the equilibrium in OLG models, Diamond [1965] model represents the benchmark model with separable preferences. de la Croix [1996] prove that the optimal solution in the Diamond [1965] model is always characterized by monotonic convergence to the steady state. Michel and Venditti [1997] provide sufficient conditions for stability of the equilibrium in an OLG model with separable preferences across generations only and prove that the optimal solution may be oscillating and optimal cycles may exist. de la Croix and Michel [1999] provide sufficient conditions for existence and uniqueness of the equilibrium in an OLG model with separable preferences within generations only and prove that the optimal solution may display damped oscillations even when the social planner does not discount the utility of future generations (golden rule case).

The contribution of this paper is to provide sufficient conditions for existence and stability of the steady state equilibrium in an OLG model in which preferences are non-separable *both* across and within generations.

Since the intergenerational transmission of habits generates an intergenerational externality, both the competitive equilibrium and the social planner problems are analyzed. The paper derives conditions under which the competitive economy converges to or diverges from the non trivial steady state and shows that this steady state may display either fluctuations or explosive behavior. Then it studies the conditions under which the optimal solution is stable. Under the assumption of intergenerational and intragenerational spillovers, convergence of the optimal solution to the non trivial steady state is not always assured. The optimal solution may display either locally explosive dynamics or damped oscillations.

This paper shows that combining different forms of non-separable preferences is not innocuous as the dynamics of the model and the stability of the equilibrium are considerably affected.

2 The model

The model is a simple extension of the Diamond [1965] economy without outside money. At each date, the economy is populated by three generations (young, adult and old), each living for three periods. The growth rate of population is zero. The young generation has no decision to take and only inherits habits h_t from the previous adult generation according to the following equation

$$h_t = c^a_{t-1} \tag{1}$$

where c^a_{t-1} is the consumption of the adult generation at time $t-1$. The adult generation draws utility from consumption of the quantity c^a_t, given its own stock of habits h_t. When old, each agent draws utility from consumption of the quantity c^o_{t+1}, given her own past consumption c^a_t. The intertemporal utility function of each adult agent is

$$U(c^a_t, c^o_{t+1}; h_t) = u(\underbrace{c^a_t - \theta h_t}_{\text{passive effect}}) + v(\underbrace{c^o_{t+1} - \delta c^a_t}_{\text{active effect}}) \tag{2}$$

where $\theta \in (0, 1)$ measures the intensity of the intergenerational spillover effect due to the inherited habits (labeled as passive effect in equation (2)) and $\delta \in (0, 1)$ measures the intensity of the intragenerational spillover effect due to the persistence of own preferences over time (labeled as active effect in equation (2)). In other words, we assume that adult consumption at time $t-1$ determines a frame of reference against which adult individual consumption at time t is judged and that the depreciation rate of these inherited habits is so high that it no longer affects the evaluation of consumption when old. We also assume that adult consumption at time t determines a frame of reference against which old individual consumption at time $t+1$ is judged and that persistence of preferences is so high that neither young consumption at time t nor that at time $t-1$ affect in any possible way the evaluation of consumption when old.

Moreover we assume that the utility function is strictly increasing with respect to consumption and decreasing with respect to the stock h: $u_{c^a} > 0$, $v_{c^o} > 0$, $u_h < 0$, $u_{c^a c^a} < 0$, $v_{c^o c^o} < 0$, $u_{hh} < 0$ and $u_{c^a h} > 0$. The assumption $u_{c^a h} > 0$ amounts to postulating that an increase in the stock h rises the desire for consumption. We also assume that starvation is ruled out in both periods

$$\lim_{c_t^a \to 0} u_{c^a} + v_{c^a} = \lim_{c_{t+1}^o \to 0} v_{c^o} = \infty \tag{3}$$

and that the utility function is strictly concave under the following condition

$$\delta < \frac{u_{c^a}}{v_{c^o}} \tag{4}$$

Note that if preferences are separable as in Diamond [1965], $\delta = 0$ and strictly concavity is always ensured by the standard set of assumptions on marginal utility, i.e. $u_{c^a} > 0$ and $v_{c^o} > 0$. If preferences are non-separable, $\delta > 0$ and concavity is ensured only if condition (4) holds. Otherwise, the utility function is flat ($\delta = \frac{u_{c^a}}{v_{c^o}}$) or convex ($\delta > \frac{u_{c^a}}{v_{c^o}}$).

At each date a single good is produced. This good can be either consumed or accumulated as capital for future production. Production occurs through a constant returns to scale technology. Per capita output y_t is a function of capital intensity k_t

$$y_t = f(k_t) \tag{5}$$

in which $f()$ is a neoclassical production function with $f_k > 0$ and $f_k k < 0$. Assuming total depreciation of capital after one period, the resource constraint of the economy is

$$y_t = c_t^a + c_t^o + k_{t+1} \tag{6}$$

At date 0 the economy is endowed with a given quantity of capital per capita k_0 and a level of inherited habits h_0.

3 The competitive economy

The competitive behavior of firms leads to the equalization of the marginal productivity of each factor to its marginal cost:

$$R_t = f_k(k_t) \tag{7}$$

$$w_t = f(k_t) - k_t f_k(k_t) \tag{8}$$

where R_t is the interest factor paid on loans and w_t is the real wage paid to workers.

The adult generation works during the period t and sells one unit of labor inelastically at any real wage w_t, consumes the quantity c_t^a and saves s_t for the next period by holding capital

$$c_t^a = w_t - s_t \tag{9}$$

while the old generation spends all her savings s_t plus interest matured and consumes c_{t+1}^o

$$c_{t+1}^o = R_{t+1} s_t \tag{10}$$

The maximization program of each individual is thus to choose c_t^a, c_{t+1}^o in order to

$$\begin{aligned} \max_{c_t^a, c_{t+1}^o} \quad & u(c_t^a - \theta h_t) + v(c_{t+1}^o - \delta c_t^a) \\ \text{subject to} \quad & c_t^a = w_t - s_t \\ & c_{t+1}^o = R_{t+1} s_t \end{aligned}$$

where w_t, R_{t+1} and h_t are given to the agent. Assuming an interior solution, under rational expectations, the above decision problem leads to following first order condition

$$u_{c^a} - \delta v_{c^a} = R_{t+1} v_{c^o} \tag{11}$$

With respect to a standard Diamond [1965] model in which $\delta = 0$, marginal utility of the young is lower, as $v_{c^o} > 0$: in order to achieve the same level of satisfaction when old, adults need to correct their satisfaction by the (negative) habit effect.

Equation (11) allows to define the following saving function

$$s_t = s(w_t, R_{t+1}, h_t) \tag{12}$$

The partial derivative of the saving function (12) are

$$s_w = \frac{u_{c^a c^a}}{u_{c^a c^a} + R_{t+1}^2 v_{c^o c^o}} > 0 \qquad s_r = \frac{-(v_{c^o} + v_{c^o c^o} c_{t+1}^o)}{u_{c^a c^a} + R_{t+1}^2 v_{c^o c^o}} \qquad s_h = \frac{-\theta u_{c^a h}}{u_{c^a c^a} + R_{t+1}^2 v_{c^o c^o}} < 0$$

Since the utility function is concave and there is no wage income in the last period of life, savings increase with wage income. The effect of the interest rate is instead ambiguous and depends on the value of the intertemporal elasticity of substitution, $\frac{v_{c^o c^o} d_{t+1}}{v_{c^o}}$. Finally, the effect of rising inherited habits is negative: when the passive effect is low, the agent has a sober lifestyle and savings are high; when the passive effect is high, the agent spends much on consumption to maintain a life standard similar to the one of their peers and their propensity to save is low.

The equilibrium condition in the capital market implies

$$k_{t+1} = s_t \tag{13}$$

Combining equations (1), (7), (8), (12) and (13), the competitive equilibrium is defined as a sequence $\{k_t, h_t; t > 0\}$ which satisfies

$$k_{t+1} = s(f(k_t) - k_t f_k(k_t), f_k(k_{t+1}), h_t) \tag{14}$$

$$h_{t+1} = f(k_t) - k_t f_k(k_t) - s(f(k_t) - k_t f_k(k_t), f_k(k_{t+1}), h_t) \tag{15}$$

Equation (14) is the clearing condition of the asset market, given that the labor market is in equilibrium (i.e. that (8) holds). It reflects the fact that savings are to be equal to the capital stock of the next period. Equation (15) is the equation (1), given that the asset and the labor markets are in equilibrium. It appears from the system above that the equilibrium can be characterized by using the following forward dynamic planar map:

$$k = s(f(k) - kf_k(k), f_k(k), h) \tag{16}$$

$$h = f(k) - kf_k(k) - s(f(k) - kf_k(k), f_k(k), h) \tag{17}$$

Proposition 1. *A positive steady state equilibrium exists and is unique if and only if* $\det(\mathbf{I} - \mathbf{J}^{CE}) \neq 0$*, where* $\mathbf{J}^{CE}$ *is the Jacobian matrix associated to the competitive equilibrium (14)-(15) and evaluated at steady state* (k, h)*.*

Proof. See Appendix A.1. □

Stability of the steady state associated to the competitive equilibrium (14)-(15) depends on parameters θ and δ. Since for some values of these parameters the hyperbolic condition may not be satisfied, it is necessary to look for the critical value of θ and δ, $\bar{\theta}$ and $\bar{\delta}$, at which the change in trajectory takes place (bifurcation) and the fixed point becomes non-hyperbolic. Non-hyperbolicity may arise when (a) at least one eigenvalue equals to 1 or to -1 or (b) if the eigenvalues are complex and conjugate. If one of these conditions is met, linear approximation cannot be used to determine the stability of the system. Otherwise, local stability properties of the linear approximation carry over to the non-linear system.

Proposition 2. *Suppose Proposition 1 holds and suppose that* $s_h = \frac{(1-s_r f_{kk})}{k f_{kk}}$*,* $s_r > \frac{1}{f_{kk}}$ *and* $\frac{|s_h|}{1-s_r f_{kk}} < 1 + \left[1 - \frac{s_w k |f_{kk}|}{1-s_r f_{kk}}\right]$*. Then the determinant of the Jacobian matrix* $\mathbf{J}^{CE}$ *is positive and equal to* 1*, the trace* $\mathbf{T}_{\mathbf{J}}^{CE}$ *is positive and smaller than* 2*, and the discriminant* Δ *is negative.*

Proof. See Appendix A.2. □

Proposition 3. *Suppose Proposition 2 holds. Then the eigenvalues of* $\mathbf{J}^{CE}$ *are complex and conjugate and the steady state is stable if and only if* $\mathrm{mod}\,\sigma(\bar{\theta}, \bar{\delta}) < 1$ *and unstable otherwise.*

Proof. If $\mathrm{mod}\sigma(\bar{\theta}, \bar{\delta}) < 1$, the system is characterized by a spiral convergence to the steady state equilibrium; if $\mathrm{mod}\,\sigma(\bar{\theta}, \bar{\delta}) = 1$, the system exhibits a period orbit; and if $\mathrm{mod}\,\sigma(\bar{\theta}, \bar{\delta}) > 1$, the system exhibits a spiral divergence from the steady state equilibrium. The system is therefore characterized by a bifurcation identified at $(\bar{\theta}, \bar{\delta})$ which are the roots of $\det(\mathbf{J}^{CE}) = 1$. $\square$

The competitive equilibrium is thus characterized by spillovers from one generation to the next and from adulthood to old age. The main components of the intergenerational spillovers are: savings by old people and past consumption levels of the previous generation. While the process transforming the savings by the old into income for the adult displays decreasing returns to scale due to the characteristics of the production function, the process transforming past consumption of the adults into consumption of the next generation displays constant returns to scale due to the characteristics of the utility function. The intragenerational spillovers is only given by the individual past consumption that feeds individual's habits from adulthood to old age. This process displays constant returns to scale, again due to the characteristics of the utility function. Thus, even though the intergenerational bequest in terms of higher wages will not be sufficient to cover the intergenerational bequest in terms of higher inherited habits, the intragenerational spillover leaves a bequest in terms of higher persistence. The combination of the positive bequests in terms of higher wages and higher persistence is sufficient to offset the negative bequest in terms of the higher externality. This leads to an increase in saving to maintain future standards of consumption that induces an expansion. When the enrichment is strong enough, the externality has already reverted to higher levels, allowing a fall in savings and the start of a recession. As the effect due to persistence is stronger than the effect due to the externality, the model is characterized by converging cycles. Thus, the competitive equilibrium still displays fluctuations, but the bifurcation corresponds to different critical values of θ and δ. Depending on the parameters θ and δ, the economy may converge to or diverge from the steady state.

4 The optimal solution

As the inherited habits introduce an externality in the model, the decentralized equilibrium is implicitly sub-optimal compared to the equilibrium that would maximize the planner's utility. Thus, hereafter, we focus our attention to the optimal solution and consider a social planner who chooses the allocation of output in order to maximize the present discount value of current and future generations.

Assuming that the social planner's discount factor is γ, the social planner maximization program is thus to choose $\{c_t^a, c_t^o\}$ and $\{k_t, h_t\}$ in order to

$$\begin{aligned} \max_{c_t^a, c_t^o, k_t, h_t} \quad & \sum_{t=0}^{\infty} \gamma^t \big[u(c_t^a - \theta h_t) + \frac{1}{\gamma} v(c_t^o - \delta c_t^a) \big] \\ \text{subject to} \quad & y_t = c_t^a + c_t^o + k_{t+1} \\ & h_t = c_{t-1}^a \end{aligned} \tag{18}$$

and given k_0 and h_0.

First order conditions are:

$$u_{c^a}(c_t^a - \theta h_t) + \gamma u_h(c_{t+1}^a - \theta h_{t+1}) = \frac{1}{\gamma} v_{c^o}(c_t^o - \delta h_t) - v_h(c_{t+1}^o - \delta h_{t+1}) \tag{19}$$

$$\frac{1}{\gamma} v_{c^o}(c_t^o - \delta h_t) = v_{c^o}(c_{t+1}^o - \delta h_{t+1}) f_k(k_{t+1}) \tag{20}$$

Equation (19) is a condition for optimal intergenerational allocation of consumption between adult and old alive at the same time. Marginal utility of the adult, corrected by the social planner to internalize the taste externality, is equalized to marginal utility of the old is equal to the marginal utility of the old. Note that, due to the presence of the taste externality and contrary to the standard Diamond [1965] model, this social planner's first order condition does not respect the individual first order condition (11). Moreover, with respect to the standard Diamond [1965] model in which $\delta = \theta = 0$, marginal utility of the adult, $u_{c^a}(c_t^a - \theta h_t) + \gamma u_h(c_{t+1}^a - \theta h_{t+1})$, is lower, as $u_h < 0$, while marginal utility of the old, $\frac{1}{\gamma} v_{c^o}(c_t^o - \delta h_t) - v_h(c_{t+1}^o - \delta h_{t+1})$, is higher, as $v_h < 0$. Equation (20) sets the optimal intertemporal allocation.

The optimal equilibrium is defined as a sequence $\{c_t^a, c_t^o, k_t, h_t; t > 0\}$ which satisfies equations (19), (20), (6) and (1) simultaneously:

$$u_{c^a}(c_t^a - \theta h_t) + \gamma u_h(c_{t+1}^a - \theta h_{t+1}) = \frac{1}{\gamma} v_{c^o}(c_t^o - \delta h_t) - v_h(c_{t+1}^o - \delta h_{t+1}) \tag{21}$$

$$\frac{1}{\gamma} v_{c^o}(c_t^o - \delta h_t) = v_{c^o}(c_{t+1}^o - \delta h_{t+1}) f_k(k_{t+1}) \tag{22}$$

$$h_t = c_{t-1}^a \tag{23}$$

$$k_{t+1} = f(k_t) - c_t^a - c_t^o \tag{24}$$

It appears from the system above that the steady state (c^a, c^o, k, h) of this optimal economy is defined by

$$u_{c^a}(c^a - \theta h) + \gamma u_h(c^a - \theta h) = \frac{1}{\gamma} v_{c^o}(c^o - \delta h) - v_h(c^o - \delta h) \tag{25}$$

$$\frac{1}{\gamma} = f_k(k) \tag{26}$$

$$h = c^a \tag{27}$$

$$k = f(k) - c^a - c^o \tag{28}$$

Equation (25) shows that in an economy with passive habits, the marginal utility of the adult is lower than the corresponding marginal utility in the standard Diamond [1965] model: the inheritance represents a benchmark from which individuals want to depart. Even the marginal utility of the old is higher than the corresponding marginal utility in the standard Diamond [1965] model: the same interpretation carries on. Once the externality associated with parents' habits is internalized, persistence affects marginal utility in the same way as the externality: they both induce consumers to save. Equation (26) is the modified golden rule: the introduction of intergenerational and intragenerational spillovers does not modify the optimal steady-state stock of capital which remains fixed at the modified golden rule level. Equations (27) and (28) have been already discussed in the paper.

Proposition 4. *A positive steady state equilibrium exists and is unique if and only if* $\det(\mathbf{I} - \mathbf{J}^{SO}) \neq 0$*, where* $\mathbf{J}^{SO}$ *is the Jacobian matrix associated to the optimal equilibrium (21)-(24) and evaluated at steady state* (c^a, c^o, k, h)*.*

Proof. See Appendix A.3. □

Proposition 5. *Assume that k and h are state variables and that c^a and c^o are jump variables. Locally explosive dynamics is possible, depending on the sign of the trace $\mathbf{T}_{\mathbf{J}}^{SO}$ and of the element Z of the Jacobian matrix $\mathbf{J}^{SO}$. If $\Delta \geq 0$, the eigenvalues are real and local dynamics is either explosive or monotonic. If $\Delta < 0$, the eigenvalues are complex and conjugate and local dynamics displays either explosive or damped oscillation.*

Proof. See Appendix A.4. □

The above proposition identifies all possible dynamics of the optimal steady state equilibrium. Under the assumption that the trace $\mathbf{T}_{\mathbf{J}}^{SO}$ and the element Z are both positive, locally explosive dynamics is identified by an unstable node if the eigenvalues are real and by an unstable focus if the eigenvalues are complex and conjugate. Under the assumption that $\mathbf{T}_{\mathbf{J}}^{SO}$ and Z are both negative, the optimal solution is a stable saddle point, only if the constraints on the elements of the Jacobian matrix $\mathbf{J}^{SO}$ respect the condition on negativity of the trace. Under the assumption that $\mathbf{T}_{\mathbf{J}}^{SO}$ and Z have opposite sign, the optimal solution may be either stable or unstable: if stable, dynamics displays damping oscillation to the steady state; if unstable, locally explosive dynamics occurs when the constraints on the elements of the matrix $\mathbf{J}^{SO}$ do not respect the condition on negativity of the trace.

The stability of the optimal steady state equilibrium depends on the assumption that habits are transmitted both across and within generations, assumption that affects the sign of the trace $\mathbf{T}_{\mathbf{J}}^{SO}$ and of element Z. Monotonic convergence to the optimal steady state equilibrium is ensured only under the assumption that the stock of inherited habits does not persist into their old age, i.e. only if $\delta = 0$ as in the standard Diamond [1965] model. Contrary to the competitive equilibrium, the optimal solution is only characterized by a positive intragenerational spillover: savings by the old, that directly finance the capital stock required for production in the next period and indirectly sustain wages of the adult. The intergenerational spillover due to habits is *a priori* internalized by the social planner in the maximization problem (18). As in the competitive equilibrium, the process transforming the savings by the old into income for the adult displays decreasing returns to scale, due to the characteristics of the production function. However, the intergenerational bequest in terms of higher wages does not interact with any other spillover. The intergenerational bequest in terms of higher wages will lead to a constant increase in saving that induces a permanent expansion. The model might thus be characterized by a diverging explosive dynamics.

5 Numerical example

Following Ferson and Constantinides [1991], we now assume that the utility function is logarithmic, $\ln(c_t^a - \theta h_t) + \beta \ln(c_{t+1}^o - \delta c_t^a)$, and that the production function is Cobb-Douglas, $y_t = k_t^\alpha$.

The steady state relative to the competitive equilibrium (14)-(15) becomes

$$\begin{aligned} k &= (\Phi + \Xi)^{-\frac{1}{1-\alpha}} \\ h &= \Omega \left[\frac{(\Phi + \Xi)^{\frac{1-2\alpha}{1-\alpha}}}{\alpha(\Phi + \Xi) + \delta} \right] \end{aligned}$$

in which

$$\begin{aligned} \Phi &\equiv \frac{1}{2}[\alpha - (1-\alpha)\delta] \left[\frac{1 + \beta(1-\theta)}{\alpha(1-\alpha)\beta(1-\theta)} \right] \\ \Xi &\equiv \frac{1}{2} \left[\frac{1 + \beta(1-\theta)}{\alpha(1-\alpha)\beta(1-\theta)} \right] \sqrt{[\alpha - (1-\alpha)\delta]^2 + 4\frac{\alpha(1-\alpha)\beta\delta(1-\theta)}{[1 + \beta(1-\theta)]}} \\ \Omega &\equiv \frac{\alpha(1-\alpha)}{1 + \beta(1-\theta)} \end{aligned}$$

Appendix A.5 provides a detailed proof of the above closed form solutions.

We then use market clearing conditions (7) and (8), resource constraint (6) and individual constraints (9) and (10) to derive steady state values of output, real wages, interest factor, consumption of the adult and of the old, respectively:

$$
\begin{aligned}
y &= (\Phi+\Xi)^{-\frac{\alpha}{1-\alpha}} \\
w &= (1-\alpha)(\Phi+\Xi)^{-\frac{\alpha}{1-\alpha}} \\
R &= \alpha(\Phi+\Xi) \\
c^a &= (1-\alpha)(\Phi+\Xi)^{-\frac{\alpha}{1-\alpha}} - (\Phi+\Xi)^{-\frac{1}{1-\alpha}} \\
c^o &= \alpha(\Phi+\Xi)^{-\frac{\alpha}{1-\alpha}}
\end{aligned}
$$

We then assign values to the relevant parameters of the model, i.e. α, β, θ and δ, and we make comparisons with the standard Diamond [1965] model.

< Table 1 about here >

If agents assign the same value to θ and δ, steady state capital is systematically higher than steady state capital in Diamond [1965]; steady state stock of habits is obviously higher as steady state stock of habits is zero in Diamond [1965]; steady state adult consumption is systematically lower than steady state adult consumption in Diamond [1965]; steady state old consumption is systematically higher than steady state old consumption in Diamond [1965]. If agents assign to intensity of the intergenerational spillover a weight lower than the one assigned to intensity of the intragenerational spillover, steady state capital is lower than steady state capital in Diamond [1965] until $\theta = \delta$, then it is higher; steady state stock of habits is increasing until $\theta = \delta$, then decreasing; steady state old consumption is lower than steady state old consumption in Diamond [1965] until $\theta = \delta$, then it is higher. The opposite is true if agents assign to intensity of the intergenerational spillover a weight higher than the one assigned to intensity of the intragenerational spillover.

< Table 2 about here >

Among all possible values of θ and δ, we are able to identify the values at which bifurcation occurs, i.e. $\bar{\theta} = 0.87286299$ and $\bar{\delta} = 0.12713701$, given the assigned values of $\alpha = 0.3$ and $\beta = 0.9$. At these specific values, $\det(\mathbf{J}^{CE})$ is equal to 1, the trace $\mathbf{T}_{\mathbf{J}}^{CE}$ is positive and smaller than 2 and the discriminant Δ of matrix $\mathbf{J}^{CE}$ is negative. Therefore the eigenvalues of matrix $\mathbf{J}^{CE}$ are complex and conjugate and equal to $(-0.82777345 \pm 0.56106249i)$.

For $\bar{\theta} = 0.87286299$ and $\bar{\delta} = 0.12713701$, the system is characterized by a spiral convergence to the steady state equilibrium if $\bmod \sigma(0.87286299; 0.12713701) < 1$; it exhibits a period orbit if $\bmod \sigma(0.87286299; 0.12713701) = 1$ and a spiral divergence from the steady state equilibrium if $\bmod \sigma(0.87286299; 0.12713701) > 1$.

Analogously, the steady state relative to the optimal equilibrium (21)-(24) becomes

$$\begin{aligned}
c^a &= h = (1-\alpha)\left[\frac{\Psi}{\Psi(1-\alpha)-\alpha}\right]^{-\frac{\alpha}{1-\alpha}} - \left[\frac{\Psi}{\Psi(1-\alpha)-\alpha}\right]^{-\frac{1}{1-\alpha}} \\
c^o &= \Psi\left\{(1-\alpha)\left[\frac{\Psi}{\Psi(1-\alpha)-\alpha}\right]^{-\frac{\alpha}{1-\alpha}} - \left[\frac{\Psi}{\Psi(1-\alpha)-\alpha}\right]^{-\frac{1}{1-\alpha}}\right\} \\
k &= \left[\frac{\Psi}{\Psi(1-\alpha)-\alpha}\right]^{-\frac{1}{1-\alpha}}
\end{aligned}$$

in which $\Psi \equiv \frac{\beta(\frac{1}{\gamma}+\delta)(1-\theta)}{1-\gamma\theta} + \delta$. Appendix A.6 provides a detailed proof of the above closed form solutions.

We then derive the steady state values of output, real wages and interest factor respectively:

$$\begin{aligned}
y &= \left[\frac{\Psi}{\Psi(1-\alpha)-\alpha}\right]^{-\frac{\alpha}{1-\alpha}} \\
w &= (1-\alpha)\left[\frac{\Psi}{\Psi(1-\alpha)-\alpha}\right]^{-\frac{\alpha}{1-\alpha}} \\
R &= \alpha\left[\frac{\Psi}{\Psi(1-\alpha)-\alpha}\right]
\end{aligned}$$

We then assign values to the relevant parameters of the model, i.e. α, β, θ, δ and γ, and we make comparisons with the standard Diamond [1965] model.

< Table 3 about here >

If agents assign the same value to θ and δ and the social planner's discount factor $\gamma = 0.5$, steady state capital is generally higher than steady state capital in Diamond [1965]; steady state adult consumption is generally lower than steady state adult consumption in Diamond [1965]; steady state old consumption is generally higher than steady state old consumption in Diamond [1965]. If agents assign to intensity of the intergenerational spillover a weight lower than the one assigned to intensity of the intragenerational spillover, steady state capital is lower than steady state capital in Diamond [1965] until $\theta = \delta$, then it is higher; steady state adult consumption is higher than steady state adult consumption in Diamond [1965] until $\theta = \delta$, then it is lower; steady state old consumption is lower than steady state old consumption in Diamond [1965] until $\theta = \delta$, then it is higher. The opposite is true if agents assign to intensity of the intergenerational spillover a weight higher than the one assigned to intensity of the intragenerational spillover.

< Table 4 about here >

When the social planner's discount factor is very high, $\gamma = 0.99$, steady state capital is systematically higher than steady state capital in Diamond [1965]; steady state adult consumption is systematically lower than steady state adult consumption in Diamond [1965]; steady state old consumption is systematically higher than steady state old consumption in Diamond [1965]. When $\gamma = 0.99$, if θ increases independently of δ, steady state capital is systematically increasing, steady state adult consumption is systematically decreasing and steady state old consumption is systematically increasing. The opposite is true if θ decreases.

< Table 5 about here >

< Table 6 about here >

When $\bar{\theta} = 0.87286299$ and $\bar{\delta} = 0.12713701$, given the assigned values of $\alpha = 0.3$, $\beta = 0.9$ and $\gamma = 0.5$ or 0.99, $\det(\mathbf{J}^{SO})$ is equal to 4 or 1.020304, the trace $\mathbf{T}_{\mathbf{J}}^{SO}$ is equal to 0.7272058 or 0.2863283 and the element Z is equal to 4.727206 or 2.30653, respectively. Therefore the eigenvalues of the characteristic polynomial P associated to the Jacobian matrix $\mathbf{J}^{SO}$ are complex and conjugate and equal to $(0.13362 \pm 1.06966i)$ and $(0.22998 \pm 1.84102i)$ when $\gamma = 0.5$ and to $(0.05376 \pm 0.77746i)$ and $(0.08941 \pm 1.29305i)$ when $\gamma = 0.99$. We can therefore conclude that the dynamics of the social optimum is characterized by a locally unstable focus.

6 Conclusions

This paper derives sufficient conditions for existence of a steady state equilibrium in an overlapping generations model with non-separable preferences and analyzes the implications of non-separable preferences for the local stability of the steady state equilibrium.

It derives the conditions for existence and stability of the equilibrium in a competitive setting and shows that the competitive economy may display either fluctuations or explosive behavior. Then it studies the conditions for the existence and stability of optimal equilibrium and it proves that the optimal solution may display damped oscillations or locally explosive dynamics. This result depends on the assumption that habits are transmitted from one generation to the next one and from adulthood to old age.

This paper shows that combining different forms of non-separable preferences is not innocuous: when we introduce persistence of individual tastes in the contest of an OLG model in which habits are inherited, dynamics of the model and stability of the equilibrium are dramatically affected. The results presented in this paper are therefore fundamental to understanding the mechanisms underneath models with habit formation and habit persistence, as habits seem to play a significant role in many aspects of economic theory.

A Appendices

A.1 Proof of Proposition 1

First, linearize the non-linear dynamic system (14)-(15) around the steady state (k, h)

$$\begin{aligned} \mathrm{d}k_{t+1} &= s_w\left[-kf_{kk}\right]\mathrm{d}k_t + s_r f_{kk}\mathrm{d}k_{t+1} + s_h\mathrm{d}h_t \\ \mathrm{d}h_{t+1} &= -kf_{kk}\mathrm{d}k_t - s_w\left[-kf_{kk}\right]\mathrm{d}k_t - s_r f_{kk}\mathrm{d}k_{t+1} - s_h\mathrm{d}h_t \end{aligned}$$

Then solve the first equation by $\mathrm{d}k_{t+1}$

$$\mathrm{d}k_{t+1} = \frac{1}{1 - s_r f_{kk}}\left[-s_w k f_{kk}\mathrm{d}k_t + s_h\mathrm{d}h_t\right]$$

and substitute the solution into the second equation of the above system

$$\mathrm{d}h_{t+1} = \frac{1}{1 - s_r f_{kk}}\left[\left(s_w - 1 + s_r f_{kk}\right)kf_{kk}\mathrm{d}k_t - s_h\mathrm{d}h_t\right]$$

The linearized system (14)-(15) around the steady state (k, h) is therefore

$$\begin{bmatrix} \mathrm{d}k_{t+1} \\ \mathrm{d}h_{t+1} \end{bmatrix} = \frac{1}{1 - s_r f_{kk}} \begin{bmatrix} -s_w k f_{kk} & s_h \\ \left(s_w - 1 + s_r f_{kk}\right)kf_{kk} & -s_h \end{bmatrix} \begin{bmatrix} \mathrm{d}k_t \\ \mathrm{d}h_t \end{bmatrix}$$

in which

$$\mathbf{J}^{CE} = \begin{bmatrix} \frac{-s_w k f_{kk}}{1 - s_r f_{kk}} & \frac{s_h}{1 - s_r f_{kk}} \\ \frac{(s_w - 1 + s_r f_{kk})kf_{kk}}{1 - s_r f_{kk}} & \frac{-s_h}{1 - s_r f_{kk}} \end{bmatrix}$$

is the Jacobian matrix evaluated at steady state (k, h). It is immediate to show that the matrix $\mathbf{I} - \mathbf{J}^{CE}$ is equal to

$$\left[\mathbf{I} - \mathbf{J}^{CE}\right] = \begin{bmatrix} 1 + \frac{s_w k f_{kk}}{1 - s_r f_{kk}} & -\frac{s_h}{1 - s_r f_{kk}} \\ -\frac{(s_w - 1 + s_r f_{kk})kf_{kk}}{1 - s_r f_{kk}} & 1 + \frac{s_h}{1 - s_r f_{kk}} \end{bmatrix}$$

and that its determinant is

$$\begin{aligned}\det\left(\mathbf{I}-\mathbf{J}^{CE}\right) &= 1+\frac{s_w k f_{kk}}{1-s_r f_{kk}}+\frac{s_h}{1-s_r f_{kk}}+\frac{s_w k f_{kk}-\left(s_w-1+s_r f_{kk}\right) k f_{kk}}{\left(1-s_r f_{kk}\right)^2} s_h = \\ &= \frac{1-s_r f_{kk}+s_h+\left(s_w+s_h\right) k f_{kk}}{1-s_r f_{kk}} \neq 0\end{aligned}$$

under regularity conditions on the utility function and on the production function, given partial derivatives of the saving function and equations (7) and (8).

A.2 Proof of Proposition 2

The determinant of the matrix $\mathbf{J}^{CE}$ is equal to

$$\det\left(\mathbf{J}^{CE}\right)=\frac{s_h s_w k f_{kk}}{\left(1-s_r f_{kk}\right)^2}-\frac{s_h\left(s_w-1+s_r f_{kk}\right) k f_{kk}}{\left(1-s_r f_{kk}\right)^2}$$

The trace is equal to

$$\mathbf{T}_{\mathbf{J}}^{CE}=-\frac{s_w k f_{kk}}{1-s_r f_{kk}}-\frac{s_h}{1-s_r f_{kk}}$$

The discriminant is defined as

$$\Delta=\left(\mathbf{T}_{\mathbf{J}}^{CE}\right)^2-4 \det \mathbf{J}^{CE}$$

Therefore, under the following conditions

$$\begin{aligned} s_h &= \frac{\left(1-s_r f_{kk}\right)}{k f_{kk}} \\ s_r &> \frac{1}{f_{kk}} \\ \frac{\left|s_h\right|}{1-s_r f_{kk}} &< 1+\left[1-\frac{s_w k\left|f_{kk}\right|}{1-s_r f_{kk}}\right]\end{aligned}$$

it is immediate to prove that the determinant is equal to 1

$$\det\left(\mathbf{J}^{CE}\right)=\frac{s_h k f_{kk}}{1-s_r f_{kk}}=1 \tag{A.1}$$

and that the trace becomes in absolute values smaller than 2

$$\mathbf{T}_{\mathbf{J}}^{CE} = \frac{s_w k\,|f_{kk}|}{1 - s_r f_{kk}} + \frac{|s_h|}{1 - s_r f_{kk}} < 2 \tag{A.2}$$

Since

$$\mathbf{T}_{\mathbf{J}}^{CE} < 2 \Rightarrow \left(\mathbf{T}_{\mathbf{J}}^{CE}\right)^2 < 4 \tag{A.3}$$

the discriminant is negative

$$\Delta = \left[\frac{s_w k\,|f_{kk}|}{1 - s_r f_{kk}} + \frac{|s_h|}{1 - s_r f_{kk}}\right]^2 - 4 < 0 \tag{A.4}$$

A.3 Proof of Proposition 4

First, linearize the non linear dynamic system (21)-(24) around the steady state (25)-(28):

$$\begin{aligned}
u_{c^a c^a}\mathrm{d}c_t^a + u_{c^a h}\mathrm{d}h_t + \gamma u_{c^a h}\mathrm{d}c_{t+1}^a + \gamma u_{hh}\mathrm{d}h_{t+1} &= \frac{1}{\gamma}v_{c^o c^o}\mathrm{d}c_t^o + \frac{1}{\gamma}v_{c^o h}\mathrm{d}h_t - v_{c^o h}\mathrm{d}c_{t+1}^o - v_{hh}\mathrm{d}h_{t+1} \\
\frac{1}{\gamma}v_{c^o c^o}\mathrm{d}c_t^o + \frac{1}{\gamma}v_{c^o h}\mathrm{d}h_t - v_{c^o c^o}f_k(k_{t+1})\mathrm{d}c_{t+1}^o &= v_{c^o h}f_k(k_{t+1})\mathrm{d}h_{t+1} + v_{c^o}f_{kk}(k_{t+1})\mathrm{d}k_{t+1} \\
\mathrm{d}h_{t+1} &= \mathrm{d}c_t^a \\
\mathrm{d}k_{t+1} &= f_k(k_t)\mathrm{d}k_t - \mathrm{d}c_t^a - \mathrm{d}c_t^o
\end{aligned}$$

Then solve the first equation by $\mathrm{d}c_{t+1}^a$ and the second by $\mathrm{d}c_{t+1}^o$, using the third and the fourth equations:

$$\begin{aligned}
\mathrm{d}c^a_{t+1} &= \left[\frac{v_{c^oc^o}}{\gamma^2 u_{c^ah}} - \frac{v_{c^oh}}{\gamma u_{c^ah}}\left(\frac{1}{\gamma f_k(k_{t+1})} + \frac{v_{c^o} f_{kk}(k_{t+1})}{v_{c^oc^o} f_k(k_{t+1})}\right)\right]\mathrm{d}c^o_t + \\
&+ \left[-\frac{1}{\gamma} + \frac{v_{c^oh}}{\gamma^2 u_{c^ah}} - \frac{v^2_{c^oh}}{\gamma^2 u_{c^ah} v_{c^oc^o} f_k(k_{t+1})}\right]\mathrm{d}h_t + \\
&+ \left[-\frac{u_{c^ac^a}}{\gamma u_{c^ah}} - \frac{\gamma u_{hh}}{\gamma u_{c^ah}} - \frac{v_{hh}}{\gamma u_{c^ah}} - \frac{v_{c^oh}}{\gamma u_{c^ah}}\left(\frac{v_{c^o} f_{kk}(k_{t+1})}{v_{c^oc^o} f_k(k_{t+1})} + \right.\right. \\
&- \left.\left.\frac{v_{c^oh}}{v_{c^oc^o}}\right)\right]\mathrm{d}c^a_t + \frac{v_{c^oh}}{\gamma u_{c^ah}}\left[\frac{v_{c^o} f_{kk}(k_{t+1})}{v_{c^oc^o} f_k(k_{t+1})} f_k(k_t)\right]\mathrm{d}k_t \\
\mathrm{d}c^o_{t+1} &= \left[\frac{v_{c^o} f_{kk}(k_{t+1})}{v_{c^oc^o} f_k(k_{t+1})} + \frac{1}{\gamma f_k(k_{t+1})}\right]\mathrm{d}c^o_t + \frac{v_{c^oh}}{\gamma v_{c^oc^o} f_k(k_{t+1})}\mathrm{d}h_t + \\
&+ \left[\frac{v_{c^o} f_{kk}(k_{t+1})}{v_{c^oc^o} f_k(k_{t+1})} - \frac{v_{c^oh}}{v_{c^oc^o}}\right]\mathrm{d}c^a_t - \frac{v_{c^o} f_{kk}(k_{t+1})}{v_{c^oc^o} f_k(k_{t+1})} f_k(k_t)\mathrm{d}k_t \\
\mathrm{d}h_{t+1} &= \mathrm{d}c^a_t \\
\mathrm{d}k_{t+1} &= f_k(k_t)\mathrm{d}k_t - \mathrm{d}c^a_t - \mathrm{d}c^o_t
\end{aligned}$$

The linearized system (21)-(24) around the steady state (25)-(28) is therefore

$$\begin{bmatrix} \mathrm{d}h_{t+1} \\ \mathrm{d}c^a_{t+1} \\ \mathrm{d}k_{t+1} \\ \mathrm{d}c^o_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{[B(1-\gamma E)-1]}{\gamma} & A - B(D-E) & \frac{DB}{\gamma} & C - B(1+D) \\ 0 & -1 & \frac{1}{\gamma} & -1 \\ E & D-E & -\frac{D}{\gamma} & 1+D \end{bmatrix} \begin{bmatrix} \mathrm{d}h_t \\ \mathrm{d}c^a_t \\ \mathrm{d}k_t \\ \mathrm{d}c^o_t \end{bmatrix}$$

in which

$$\begin{aligned}
A &\equiv -\frac{u_{c^ac^a} + \gamma u_{hh} + v_{hh}}{\gamma u_{c^ah}} > 0 \\
B &\equiv \frac{v_{c^oh}}{\gamma u_{c^ah}} > 0 \\
C &\equiv \frac{v_{c^oc^o}}{\gamma^2 u_{c^ah}} < 0 \\
D &\equiv \frac{\gamma v_{c^o} f_{kk}}{v_{c^oc^o}} > 0 \\
E &\equiv \frac{v_{c^oh}}{v_{c^oc^o}} < 0
\end{aligned}$$

under the assumption that in steady state $f_k(k_{t+1}) = f_k(k_t) = \gamma^{-1}$.

As the Jacobian matrix evaluated at steady state (c^a, c^o, k, h) is

$$\mathbf{J}^{SO}=\begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{[B(1-\gamma E)-1]}{\gamma} & A-B(D-E) & \frac{DB}{\gamma} & C-B(1+D) \\ 0 & -1 & \frac{1}{\gamma} & -1 \\ E & D-E & -\frac{D}{\gamma} & 1+D \end{bmatrix}$$

it is immediate to show that the matrix $\mathbf{I}-\mathbf{J}^{SO}$ is equal to

$$\mathbf{I}-\mathbf{J}^{SO}=\begin{bmatrix} 1 & -1 & 0 & 0 \\ -\frac{[B(1-\gamma E)-1]}{\gamma} & 1-A+B(D-E) & -\frac{DB}{\gamma} & -C+B(1+D) \\ 0 & 1 & -\frac{1}{\gamma} & 1 \\ -E & -D+E & \frac{D}{\gamma} & -1-D \end{bmatrix}$$

and that its determinant is

$$\begin{aligned}\det(\mathbf{I}-\mathbf{J}^{SO}) &= 1-[A-B(D-E)+\gamma^{-1}+(1+D)]+\frac{1-B+\gamma CE}{\gamma^2} \\ &= -A+B(D-E)-\gamma^{-1}-D+\frac{1-B+\gamma CE}{\gamma^2} \\ &= \frac{u_{c^a c^a}+\gamma u_{hh}+v_{hh}}{\gamma u_{c^a h}}-\frac{1}{\gamma}(1-\frac{1}{\gamma})+ \\ &\quad +\frac{v_{c^o h}}{\gamma u_{c^a h}}(\frac{\gamma v_{c^o} f_{kk}-v_{c^o h}}{v_{c^o c^o}})-\frac{\gamma v_{c^o} f_{kk}}{v_{c^o c^o}}\neq 0\end{aligned}$$

under the assumptions that the utility function is concave, that the production function is neoclassical, that equations (25)-(28) hold and that conditions (27) and (28) are met.

A.4 Proof of Proposition 5

The characteristic polynomial P in the eigenvalues σ associated to the Jacobian matrix $\mathbf{J}^{SO}$ evaluated at steady state (c^a, c^o, k, h) is

$$P(\sigma)=\sigma^4-\mathbf{T}_{\mathbf{J}}^{SO}\sigma^3+Z\sigma^2-\gamma^{-1}\mathbf{T}_{\mathbf{J}}^{SO}\sigma+\det\mathbf{J}^{SO}=0$$

in which

$$
\begin{aligned}
\det(\mathbf{J}^{SO}) &= \frac{1-B+\gamma CE}{\gamma^2} = \gamma^{-2} > 0 \\
\mathbf{T}_{\mathbf{J}}^{SO} &= 1+\gamma^{-1}+A-B(D-E)+D \gtrless 0 \\
&\quad \text{if} \quad 1+\gamma^{-1}+A+D \gtrless B(D-E) \\
Z &= 2\gamma^{-1}+(1+\gamma^{-1}+A-B(D-E)+D) \gtrless 0 \\
&\quad \text{if} \quad 1+3\gamma^{-1}+A+D \gtrless B(D-E)
\end{aligned}
$$

In order to study the polynomial P, factorize the polynomial P into

$$P(\sigma) = (\sigma-\sigma_1)(\sigma-\sigma_2)(\sigma-\sigma_3)(\sigma-\sigma_4) = 0$$

which is equivalent to

$$\left(\sigma^2-\phi_1\sigma+\frac{1}{\gamma}\right)\left(\sigma^2-\phi_2\sigma+\frac{1}{\gamma}\right) = 0$$

in which $\phi_1 \equiv \sigma_1+\sigma_2$ and $\phi_2 \equiv \sigma_3+\sigma_4$. Then analyze all possible scenarios, due to the sign's ambiguity of the trace $\mathbf{T}_{\mathbf{J}}^{SO}$ and of element Z.

First, assume that $\mathbf{T}_{\mathbf{J}}^{SO}$ and Z are both positive and analyze the two possible cases:

1. $\Delta \equiv \phi_i^2 - 4\gamma^{-1} \geqslant 0, i = 1,2$. The four eigenvalues are real and they can be:

   (a) four negative roots. This case implies that $\phi_1+\phi_2 < 0$ and $\phi_1 \cdot \phi_2 > 0$. This case is excluded as it violates $\mathbf{T}_{\mathbf{J}}^{SO} > 0$.

   (b) two negative and two positive roots. This case implies that $\phi_1+\phi_2 \gtrless 0$ and $\phi_1 \cdot \phi_2 < 0$. This case is excluded as it violates $Z > 0$.

   (c) four positive roots. This case implies that $\phi_1+\phi_2 > 0$ and $\phi_1 \cdot \phi_2 > 0$. This case is accepted as it respects both conditions on $\mathbf{T}_{\mathbf{J}}^{SO}$ and Z.

2. $\Delta \equiv \phi_i^2 - 4\gamma^{-1} < 0, i = 1,2$. Look at the real parts only. Since the real part $a = -\frac{1}{2}\phi_i \neq 0, i = 1,2$, the eigenvalues are complex and conjugate and they can be:

(a) four negative roots. This case implies that $\phi_1 + \phi_2 < 0$ and $\phi_1 \cdot \phi_2 > 0$. This case is excluded as it violates $\mathbf{T}_{\mathbf{J}}^{SO} > 0$.

(b) two negative and two positive roots. This case implies that $\phi_1 + \phi_2 \gtreqless 0$ and $\phi_1 \cdot \phi_2 < 0$. This case is excluded as it violates $Z > 0$.

(c) four positive roots. This case implies that $\phi_1 + \phi_2 > 0$ and $\phi_1 \cdot \phi_2 > 0$. This case is accepted as it respects both conditions on $\mathbf{T}_{\mathbf{J}}^{SO}$ and Z.

Under the assumption that $\mathbf{T}_{\mathbf{J}}^{SO}$ and Z are both positive, the only admissible case is (c). It identifies an unstable node if the eigenvalues are real and an unstable focus if the eigenvalues are complex and conjugate. Locally explosive dynamics is highly likely.

Then, assume that $\mathbf{T}_{\mathbf{J}}^{SO}$ and Z are both negative and analyze the two possible cases:

1. $\Delta \equiv \phi_i^2 - 4\gamma^{-1} \geqslant 0, i = 1, 2$. The four eigenvalues are real and they can be:

   (a) four positive roots. This case implies that $\phi_1 + \phi_2 > 0$ and $\phi_1 \cdot \phi_2 > 0$. This case is excluded as it violates both conditions on $\mathbf{T}_{\mathbf{J}}^{SO}$ and Z.

   (b) two negative and two positive roots. This case implies that $\phi_1 + \phi_2 \gtreqless 0$ and $\phi_1 \cdot \phi_2 < 0$. This case is admissible only if $\phi_1 + \phi_2 < 0$ as $\mathbf{T}_{\mathbf{J}}^{SO} < 0$.

   (c) four negative roots. This case implies that $\phi_1 + \phi_2 < 0$ and $\phi_1 \cdot \phi_2 > 0$. This case is excluded as it violates condition on $Z > 0$.

2. $\Delta \equiv \phi_i^2 - 4\gamma^{-1} < 0, i = 1, 2$. Look at the real parts only. Since the real part $a = -\frac{1}{2}\phi_i \neq 0, i = 1, 2$, the eigenvalues are complex and conjugate and they can be:

   (a) four positive roots. This case implies that $\phi_1 + \phi_2 > 0$ and $\phi_1 \cdot \phi_2 > 0$. This case is excluded as it violates both conditions on $\mathbf{T}_{\mathbf{J}}^{SO}$ and Z.

   (b) two negative and two positive roots. This case implies that $\phi_1 + \phi_2 \gtreqless 0$ and $\phi_1 \cdot \phi_2 < 0$. This case is admissible only if $\phi_1 + \phi_2 < 0$ as $\mathbf{T}_{\mathbf{J}}^{SO} < 0$.

   (c) four negative roots. This case implies that $\phi_1 + \phi_2 < 0$ and $\phi_1 \cdot \phi_2 > 0$. This case is excluded as it violates condition on $Z > 0$.

Under the assumption that $\mathbf{T}_{\mathbf{J}}^{SO}$ and Z are both negative, the only admissible case is b), but only if $\phi_1+\phi_2<0$. It identifies a stable saddle point that ensures monotonic local convergence.

Finally, assume that $\mathbf{T}_{\mathbf{J}}^{SO}$ and Z have opposite sign and distinguish two possible cases:

1. $\Delta \equiv \phi_i^2 - 4\gamma^{-1} \geqslant 0, i = 1, 2$. The four eigenvalues are real and they can be:

   (a) four positive roots. This case implies that $\phi_1 + \phi_2 > 0$ and $\phi_1 \cdot \phi_2 > 0$. This case is excluded as it violates both conditions on $\mathbf{T}_{\mathbf{J}}^{SO}$ and Z.

   (b) two negative and two positive roots. This case implies that $\phi_1 + \phi_2 \gtreqless 0$ and $\phi_1 \cdot \phi_2 < 0$. This case is admissible only if $\phi_1 + \phi_2 > 0$ as is ensures $\mathbf{T}_{\mathbf{J}}^{SO} > 0$.

   (c) four negative roots. This case implies that $\phi_1 + \phi_2 < 0$ and $\phi_1 \cdot \phi_2 > 0$. This case is admissible only if $\mathbf{T}_{\mathbf{J}}^{SO} < 0$ and $Z > 0$.

2. $\Delta \equiv \phi_i^2 - 4\gamma^{-1} < 0, i = 1, 2$. Look at the real parts only. Since the real part $a = -\frac{1}{2}\phi_i \neq 0, i = 1, 2$, the eigenvalues are complex and conjugate and they can be:

   (a) four positive roots. This case implies that $\phi_1 + \phi_2 > 0$ and $\phi_1 \cdot \phi_2 > 0$. This case is excluded as it violates both conditions on $\mathbf{T}_{\mathbf{J}}^{SO}$ and Z.

   (b) two negative and two positive roots. This case implies that $\phi_1 + \phi_2 \gtreqless 0$ and $\phi_1 \cdot \phi_2 < 0$. This case is admissible only if $\phi_1 + \phi_2 > 0$ as it ensures $\mathbf{T}_{\mathbf{J}}^{SO} > 0$.

   (c) four negative roots. This case implies that $\phi_1 + \phi_2 < 0$ and $\phi_1 \cdot \phi_2 > 0$. This case is admissible only if $\mathbf{T}_{\mathbf{J}}^{SO} < 0$ and $Z > 0$.

Under the assumption that the trace and the element Z have opposite sign, case (b) identifies an unstable solution as $\mathbf{T}_{\mathbf{J}}^{SO} > 0$. Locally-explosive dynamics is highly likely. Case (c) identifies a stable node for real eigenvalues and a stable focus for complex and conjugate eigenvalues, and therefore it ensures damped convergence to the steady state.

A.5 Closed form solutions in the competitive equilibrium

If the utility function is logarithmic, $\ln(c_t^a - \theta h_t) + \beta \ln(c_{t+1}^o - \delta c_t^a)$, and the production function is Cobb-Douglas, $y_t = k_t^\alpha$, the competitive equilibrium (14)-(15) becomes

$$\begin{aligned} k_{t+1} &= \frac{\beta}{1+\beta}[(1-\alpha)k_t^\alpha - \theta h_t] + \frac{\delta(1-\alpha)k_t^\alpha}{(\alpha k_{t+1}^{\alpha-1}+\delta)(1+\beta)} \\ h_{t+1} &= \frac{\alpha(1-\alpha)k_t^{2\alpha-1}}{(\alpha k_{t+1}^{\alpha-1}+\delta)(1+\beta)} + \frac{\beta}{1+\beta}\theta h_t \end{aligned}$$

and the associated steady state (16)-(17) becomes

$$\begin{aligned} k &= \frac{\beta}{1+\beta}[(1-\alpha)k^\alpha - \theta h] + \frac{\delta(1-\alpha)k^\alpha}{(\alpha k^{\alpha-1}+\delta)(1+\beta)} \\ h &= \frac{\alpha(1-\alpha)k^{2\alpha-1}}{(\alpha k^{\alpha-1}+\delta)(1+\beta)} + \frac{\beta}{1+\beta}\theta h \end{aligned}$$

Derive h from the second equation of the above system and get

$$h = \alpha' \frac{k^{2\alpha-1}}{\alpha k^{\alpha-1}+\delta}$$

in which $\alpha' \equiv \frac{\alpha(1-\alpha)}{[1+\beta(1-\theta)]}$.

Substitute h back into the steady state equation relative to k and get the following second order equation in $k^{\alpha-1}$

$$\beta[\alpha'\theta - \alpha(1-\alpha)]k^{2(\alpha-1)} + (1+\beta)[\alpha-(1-\alpha)\delta]k^{\alpha-1} + (1+\beta)\delta = 0$$

Since $\alpha' \equiv \frac{\alpha(1-\alpha)}{[1+\beta(1-\theta)]}$,

$$\begin{aligned} \beta[\alpha'\theta - \alpha(1-\alpha)] &= \beta\left[\frac{\alpha(1-\alpha)}{[1+\beta(1-\theta)]}\theta - \alpha(1-\alpha)\right] \\ &= \beta\left[\frac{\alpha(1-\alpha)\theta - \alpha(1-\alpha)(1+\beta-\beta\theta)}{1+\beta(1-\theta)}\right] \\ &= \frac{\alpha(1-\alpha)\beta}{1+\beta(1-\theta)}[\theta - 1 - \beta + \beta\theta] \\ &= -\frac{\alpha(1-\alpha)\beta(1+\beta)(1-\theta)}{1+\beta(1-\theta)} \end{aligned}$$

and the second order equation in $k^{\alpha-1}$ becomes

$$
\begin{aligned}
-\frac{\alpha(1-\alpha)\beta(1+\beta)(1-\theta)}{1+\beta(1-\theta)}k^{2(\alpha-1)} + (1+\beta)[\alpha-(1-\alpha)\delta]k^{\alpha-1} + (1+\beta)\delta &= 0 \\
-\frac{\alpha(1-\alpha)\beta(1-\theta)}{[1+\beta(1-\theta)]\delta}k^{2(\alpha-1)} + \frac{[\alpha-(1-\alpha)\delta]}{\delta}k^{\alpha-1} + 1 &= 0
\end{aligned}
$$

Setting $a \equiv -\frac{\alpha(1-\alpha)\beta(1-\theta)}{[1+\beta(1-\theta)]\delta}$, $b \equiv \frac{[\alpha-(1-\alpha)\delta]}{\delta}$, $c \equiv 1$ and $k^{\alpha-1} \equiv x$, it is possible to rewrite the above second order equation in $k^{\alpha-1}$ as a quadratic equation of the type

$$
ax^2 + bx + c = 0
$$

Such equation has the following set of possible solutions

$$
x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-\frac{[\alpha-(1-\alpha)\delta]}{\delta} \pm \sqrt{\{\frac{[\alpha-(1-\alpha)\delta]}{\delta}\}^2 + 4\frac{\alpha(1-\alpha)\beta(1-\theta)}{[1+\beta(1-\theta)]\delta}}}{-2\frac{\alpha(1-\alpha)\beta(1-\theta)}{[1+\beta(1-\theta)]\delta}}
$$

Since capital cannot be an imaginary number, the discriminant

$$
\Delta \equiv \{\frac{[\alpha-(1-\alpha)\delta]}{\delta}\}^2 + 4\frac{\alpha(1-\alpha)\beta(1-\theta)}{[1+\beta(1-\theta)]\delta}
$$

must be non-negative. Since α, β, δ, $(1-\theta)$, $(1-\alpha)$ and $[1+\beta(1-\theta)]$ are all positive, the discriminant is strictly positive.

Now study the sign of the sequence $\{a, b, c\}$. By Descartes' theorem, equation $ax^2+bx+c = 0$ has a positive and a negative solution, whatever the sign of the coefficient b is. Since physical capital cannot be negative, the negative solution is excluded *a priori*. Therefore, the unique solution for equation $ax^2 + bx + c = 0$ is

$$
\begin{aligned}
x &= \frac{\frac{[\alpha-(1-\alpha)\delta]}{\delta} + \sqrt{\{\frac{[\alpha-(1-\alpha)\delta]}{\delta}\}^2 + 4\frac{\alpha(1-\alpha)\beta(1-\theta)}{[1+\beta(1-\theta)]\delta}}}{2\frac{\alpha(1-\alpha)\beta(1-\theta)}{[1+\beta(1-\theta)]\delta}} \\
&= \frac{1}{2}[\alpha-(1-\alpha)\delta][\frac{1+\beta(1-\theta)}{\alpha(1-\alpha)\beta(1-\theta)}] + \frac{1}{2}[\frac{1+\beta(1-\theta)}{\alpha(1-\alpha)\beta(1-\theta)}]\sqrt{[\alpha-(1-\alpha)\delta]^2 + 4\frac{\alpha(1-\alpha)\beta\delta(1-\theta)}{[1+\beta(1-\theta)]}}
\end{aligned}
$$

Using $k^{\alpha-1} \equiv x$, the steady state value of k is

$$k = \left\{ \frac{1}{2}[\alpha-(1-\alpha)\delta][\frac{1+\beta(1-\theta)}{\alpha(1-\alpha)\beta(1-\theta)}]+\frac{1}{2}[\frac{1+\beta(1-\theta)}{\alpha(1-\alpha)\beta(1-\theta)}]\sqrt{[\alpha-(1-\alpha)\delta]^2+4\frac{\alpha(1-\alpha)\beta\delta(1-\theta)}{[1+\beta(1-\theta)]}} \right\}^{-\frac{1}{1-\alpha}}$$

Therefore, the closed form solution of the steady state system (16)-(17) is

$$\begin{aligned} k &= \left\{ \frac{1}{2}[\alpha-(1-\alpha)\delta]\left[\frac{1+\beta(1-\theta)}{\alpha(1-\alpha)\beta(1-\theta)}\right] \right. \\ &\quad \left. + \frac{1}{2}\left[\frac{1+\beta(1-\theta)}{\alpha(1-\alpha)\beta(1-\theta)}\right]\sqrt{[\alpha-(1-\alpha)\delta]^2+4\frac{\alpha(1-\alpha)\beta\delta(1-\theta)}{[1+\beta(1-\theta)]}} \right\}^{-\frac{1}{1-\alpha}} = (\Phi+\Xi)^{-\frac{1}{1-\alpha}} \\ h &= \left[\frac{\alpha(1-\alpha)}{1+\beta(1-\delta)}\right] \\ &\quad \times \frac{\left\{\frac{1}{2}[\alpha-(1-\alpha)\delta][\frac{1+\beta(1-\theta)}{\alpha(1-\alpha)\beta(1-\theta)}]+\frac{1}{2}[\frac{1+\beta(1-\theta)}{\alpha(1-\alpha)\beta(1-\theta)}]\sqrt{[\alpha-(1-\alpha)\delta]^2+4\frac{\alpha(1-\alpha)\beta\delta(1-\theta)}{[1+\beta(1-\theta)]}}\right\}^{\frac{1-2\alpha}{1-\alpha}}}{\alpha\left\{\frac{1}{2}[\alpha-(1-\alpha)\delta][\frac{1+\beta(1-\theta)}{\alpha(1-\alpha)\beta(1-\theta)}]+\frac{1}{2}[\frac{1+\beta(1-\theta)}{\alpha(1-\alpha)\beta(1-\theta)}]\sqrt{[\alpha-(1-\alpha)\delta]^2+4\frac{\alpha(1-\alpha)\beta\delta(1-\theta)}{[1+\beta(1-\theta)]}}\right\}+\delta} \\ &= \Omega\left[\frac{(\Phi+\Xi)^{\frac{1-2\alpha}{1-\alpha}}}{\alpha(\Phi+\Xi)+\delta}\right] \end{aligned}$$

in which

$$\begin{aligned} \Phi &\equiv \frac{1}{2}[\alpha-(1-\alpha)\delta]\left[\frac{1+\beta(1-\theta)}{\alpha(1-\alpha)\beta(1-\theta)}\right] \\ \Xi &\equiv \frac{1}{2}\left[\frac{1+\beta(1-\theta)}{\alpha(1-\alpha)\beta(1-\theta)}\right]\sqrt{[\alpha-(1-\alpha)\delta]^2+4\frac{\alpha(1-\alpha)\beta\delta(1-\theta)}{[1+\beta(1-\theta)]}} \\ \Omega &\equiv \frac{\alpha(1-\alpha)}{1+\beta(1-\theta)} \end{aligned}$$

A.6 Closed form solutions in the optimal equilibrium

If the utility function is logarithmic, $\ln(c_t^a-\theta h_t)+\beta\ln(c_{t+1}^o-\delta c_t^a)$, and the production function is Cobb-Douglas, $y_t=k_t^\alpha$, the optimal equilibrium (21)-(24) becomes

$$\begin{aligned}
\frac{1}{c_t^a - \theta h_t} - \gamma\theta \frac{1}{c_{t+1}^a - \theta h_{t+1}} &= \frac{1}{\gamma}\frac{\beta}{c_t^o - \delta h_t} + \delta \frac{\beta}{c_{t+1}^o - \delta h_{t+1}} \\
\frac{1}{\gamma}\frac{\beta}{c_t^o - \delta h_t} &= \frac{\beta}{c_{t+1}^o - \delta h_{t+1}} \alpha k_{t+1}^{\alpha-1} \\
h_t &= c_{t-1}^a \\
k_{t+1} &= k_t^\alpha - c_t^a - c_t^o
\end{aligned}$$

and the associated steady state (25)-(28) becomes

$$\begin{aligned}
\frac{1}{c^a - \theta h} - \gamma\theta \frac{1}{c^a - \theta h} &= \frac{1}{\gamma}\frac{\beta}{c^o - \delta h} + \delta \frac{\beta}{c^o - \delta h} \\
\frac{1}{\gamma}\frac{\beta}{c^o - \delta h} &= \frac{\beta}{c^o - \delta h} \alpha k^{\alpha-1} \\
h &= c^a \\
k^\alpha &= k + c^a + c^o
\end{aligned}$$

Solve the first by c^o and get

$$c^o = \left[\frac{\beta(\frac{1}{\gamma} + \delta)(1-\theta)}{1-\gamma\theta} + \delta\right] c^a = \Psi c^a$$

in which

$$\Psi \equiv \frac{\beta(\frac{1}{\gamma} + \delta)(1-\theta)}{1-\gamma\theta} + \delta$$

Using the fact that in equilibrium $c^a = (1-\alpha)k^\alpha - k$, substitute c^a and c^o into the resource constraint and get

$$k^\alpha = k + c^a + c^o = (1-\alpha)(1+\Psi)k^\alpha - \Psi k$$

which can be solved by k

$$k = \left[\frac{\Psi}{\Psi(1-\alpha) - \alpha} \right]^{-\frac{1}{1-\alpha}}$$

Now use k to find steady state values of $c^a = h$ and c^o:

$$\begin{aligned} c^a &= h = (1-\alpha)\left[\frac{\Psi}{\Psi(1-\alpha) - \alpha} \right]^{-\frac{\alpha}{1-\alpha}} - \left[\frac{\Psi}{\Psi(1-\alpha) - \alpha} \right]^{-\frac{1}{1-\alpha}} \\ c^o &= \Psi\left\{ (1-\alpha)\left[\frac{\Psi}{\Psi(1-\alpha) - \alpha} \right]^{-\frac{\alpha}{1-\alpha}} - \left[\frac{\Psi}{\Psi(1-\alpha) - \alpha} \right]^{-\frac{1}{1-\alpha}} \right\} \end{aligned}$$

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Tables

Table 1: Steady state values in the competitive equilibrium

	Diamond [1965] model			**this paper**		
	$\alpha = 0.3$, $\beta = 0.9$			$\alpha = 0.3$, $\beta = 0.9$		
	$\theta = \delta = 0$	$\theta = \delta = 0.1$	$\theta = \delta = 0.3$	$\theta = \delta = 0.5$	$\theta = \delta = 0.7$	$\theta = \delta = 0.9$
k	0.2066	0.2269	0.2618	0.2842	0.2920	0.2830
h	0.0000	0.1421	0.1381	0.1342	0.1326	0.1344
y	0.6231	0.6409	0.6689	0.6856	0.6912	0.6848
w	0.4361	0.4486	0.4683	0.4799	0.4838	0.4793
R	0.9048	0.8472	0.7666	0.7238	0.7102	0.7259
c^a	0.2296	0.2217	0.2065	0.1958	0.1919	0.1963
c^o	0.1869	0.1923	0.2007	0.2057	0.2074	0.2054
	Diamond [1965] model			**this paper**		
	$\alpha = 0.3$, $\beta = 0.9$			$\alpha = 0.3$, $\beta = 0.9$		
	$\theta = \delta = 0$	$\theta = 0.1$, $\delta = 0.9$	$\theta = 0.3$, $\delta = 0.7$	$\theta = 0.5$, $\delta = 0.5$	$\theta = 0.7$, $\delta = 0.3$	$\theta = 0.9$, $\delta = 0.1$
k	0.2066	0.4311	0.3739	0.2842	0.1491	0.0238
h	0.0000	0.0876	0.1096	0.1342	0.1392	0.0666
y	0.6231	0.7769	0.7444	0.6856	0.5650	0.3259
w	0.4361	0.5438	0.5211	0.4799	0.3955	0.2281
R	0.9048	0.5407	0.5973	0.7238	1.1368	4.1058
c^a	0.2296	0.1128	0.1472	0.1958	0.2464	0.2043
c^o	0.1869	0.2331	0.2233	0.2057	0.1695	0.0978
	Diamond [1965] model			**this paper**		
	$\alpha = 0.3$, $\beta = 0.9$			$\alpha = 0.3$, $\beta = 0.9$		
	$\theta = \delta = 0$	$\theta = 0.9$, $\delta = 0.1$	$\theta = 0.7$, $\delta = 0.3$	$\theta = 0.5$, $\delta = 0.5$	$\theta = 0.3$, $\delta = 0.7$	$\theta = 0.1$, $\delta = 0.9$
k	0.2066	0.0238	0.1491	0.2842	0.3739	0.4311
h	0.0000	0.0666	0.1392	0.1342	0.1096	0.0876
y	0.6231	0.3259	0.565	0.6856	0.7444	0.7769
w	0.4361	0.2281	0.3955	0.4799	0.5211	0.5438
R	0.9048	4.1058	1.1368	0.7238	0.5973	0.5407
c^a	0.2296	0.2043	0.2464	0.1958	0.1472	0.1128
c^o	0.1869	0.0978	0.1695	0.2057	0.2233	0.2331

Table 2: Determinant, trace and discriminant of matrix $\mathbf{J}^{CE}$ with $\alpha = 0.3$ and $\beta = 0.9$

θ	δ	$\det(\mathbf{J}^{CE})$	$\mathbf{T}_{\mathbf{J}}^{CE}$	Δ
0.1	0.1	0.0301677	0.3879298	0.0298188
0.2	0.2	0.0597297	0.4713749	-0.0167243
0.3	0.3	0.0879759	0.5488476	-0.05067
0.4	0.4	0.1149739	0.621002	-0.0742521
0.5	0.5	0.1412331	0.6894475	-0.0895947
0.6	0.6	0.1674667	0.7560009	-0.0983293
0.7	0.7	0.1945311	0.822475	-0.1016595
0.8	0.8	0.2235291	0.8908297	-0.100539
0.9	0.9	0.2561193	0.9636522	-0.0958515
0.9	0.1	1.334589	1.986411	-1.392527
0.8	0.2	0.5575981	1.214424	-0.7555659
0.7	0.3	0.3146456	0.9491924	-0.3576161
0.6	0.4	0.2038696	0.7980331	-0.1786214
0.5	0.5	0.1412331	0.6894475	-0.0895947
0.4	0.6	0.0994906	0.6017719	-0.0358329
0.3	0.7	0.0681002	0.5259799	0.0042542
0.2	0.8	0.0424081	0.4575647	0.0397331
0.1	0.9	0.0201203	0.3940224	0.0747726
0.1	0.9	0.0201203	0.3940224	0.0747726
0.2	0.8	0.0424081	0.4575647	0.0397331
0.3	0.7	0.0681002	0.5259799	0.0042542
0.4	0.6	0.0994906	0.6017719	-0.0358329
0.5	0.5	0.1412331	0.6894475	-0.0895947
0.6	0.4	0.2038696	0.7980331	-0.1786214
0.7	0.3	0.3146456	0.9491924	-0.3576161
0.8	0.2	0.5575981	1.214424	-0.7555659
0.872863	**0.127137**	**1**	**1.655547**	**-1.259164**
0.9	0.1	1.334589	1.986411	-1.392527

Table 3: Steady state values in the optimal solution

	Diamond [1965] model	**this paper**				
	$\alpha = 0.3$, $\beta = 0.9$, $\gamma = 0.5$	$\alpha = 0.3$, $\beta = 0.9$, $\gamma = 0.5$				
	$\theta = \delta = 0$	$\theta = \delta = 0.1$	$\theta = \delta = 0.3$	$\theta = \delta = 0.5$	$\theta = \delta = 0.7$	$\theta = \delta = 0.9$
k	0.4074	0.4161	0.4261	0.4257	0.4095	0.3523
y	0.7638	0.7687	0.7742	0.7740	0.7650	0.7312
w	0.5347	0.5381	0.5419	0.5418	0.5355	0.5119
R	0.5625	0.5542	0.5451	0.5455	0.5604	0.6227
ca	0.1273	0.1220	0.1159	0.1161	0.1260	0.1596
co	0.2292	0.2306	0.2323	0.2322	0.2295	0.2194
	Diamond [1965] model	**this paper**				
	$\alpha = 0.3$, $\beta = 0.9$, $\gamma = 0.5$	$\alpha = 0.3$, $\beta = 0.9$, $\gamma = 0.5$				
	$\theta = \delta = 0$	$\theta = 0.1$, $\delta = 0.9$	$\theta = 0.3$, $\delta = 0.7$	$\theta = 0.5$, $\delta = 0.5$	$\theta = 0.7$, $\delta = 0.3$	$\theta = 0.9$, $\delta = 0.1$
k	0.4074	0.4948	0.4694	0.4257	0.3308	0.0048
y	0.7638	0.8097	0.7970	0.7740	0.7176	0.2014
w	0.5347	0.5668	0.5579	0.5418	0.5023	0.1410
R	0.5625	0.4910	0.5094	0.5455	0.6507	12.6207
ca	0.1273	0.0720	0.0885	0.1161	0.1715	0.1362
co	0.2292	0.2429	0.2391	0.2322	0.2153	0.0604
	Diamond [1965] model	**this paper**				
	$\alpha = 0.3$, $\beta = 0.9$, $\gamma = 0.5$	$\alpha = 0.3$, $\beta = 0.9$, $\gamma = 0.5$				
	$\theta = \delta = 0$	$\theta = 0.9$, $\delta = 0.1$	$\theta = 0.7$, $\delta = 0.3$	$\theta = 0.5$, $\delta = 0.5$	$\theta = 0.3$, $\delta = 0.7$	$\theta = 0.1$, $\delta = 0.9$
k	0.4074	0.0048	0.3308	0.4257	0.4694	0.4948
y	0.7638	0.2014	0.7176	0.7740	0.7970	0.8097
w	0.5347	0.1410	0.5023	0.5418	0.5579	0.5668
R	0.5625	12.6207	0.6507	0.5455	0.5094	0.4910
ca	0.1273	0.1362	0.1715	0.1161	0.0885	0.0720
co	0.2292	0.0604	0.2153	0.2322	0.2391	0.2429

Table 4: Steady state values in the optimal solution - continued

	Diamond [1965] model	**this paper**				
	$\alpha = 0.3$, $\beta = 0.9$, $\gamma = 0.99$	$\alpha = 0.3$, $\beta = 0.9$, $\gamma = 0.99$				
	$\theta = \delta = 0$	$\theta = \delta = 0.1$	$\theta = \delta = 0.3$	$\theta = \delta = 0.5$	$\theta = \delta = 0.7$	$\theta = \delta = 0.9$
k	0.2416	0.2963	0.3678	0.4119	0.4411	0.4580
y	0.6530	0.6942	0.7408	0.7664	0.7823	0.7911
w	0.4571	0.4860	0.5185	0.5365	0.5476	0.5538
R	0.8108	0.7030	0.6043	0.5582	0.5320	0.5182
ca	0.2155	0.1897	0.1508	0.1246	0.1065	0.0958
co	0.1959	0.2083	0.2222	0.2299	0.2347	0.2373
	Diamond [1965] model	**this paper**				
	$\alpha = 0.3$, $\beta = 0.9$, $\gamma = 0.99$	$\alpha = 0.3$, $\beta = 0.9$, $\gamma = 0.99$				
	$\theta = \delta = 0$	$\theta = 0.1$, $\delta = 0.9$	$\theta = 0.3$, $\delta = 0.7$	$\theta = 0.5$, $\delta = 0.5$	$\theta = 0.7$, $\delta = 0.3$	$\theta = 0.9$, $\delta = 0.1$
k	0.2416	0.4653	0.4431	0.4119	0.3645	0.2748
y	0.6530	0.7949	0.7833	0.7664	0.7388	0.6788
w	0.4571	0.5565	0.5483	0.5365	0.5172	0.4751
R	0.8108	0.5125	0.5304	0.5582	0.6080	0.7409
ca	0.2155	0.0911	0.1053	0.1246	0.1526	0.2003
co	0.1959	0.2385	0.2350	0.2299	0.2216	0.2036
	Diamond [1965] model	**this paper**				
	$\alpha = 0.3$, $\beta = 0.9$, $\gamma = 0.99$	$\alpha = 0.3$, $\beta = 0.9$, $\gamma = 0.99$				
	$\theta = \delta = 0$	$\theta = 0.9$, $\delta = 0.1$	$\theta = 0.7$, $\delta = 0.3$	$\theta = 0.5$, $\delta = 0.5$	$\theta = 0.3$, $\delta = 0.7$	$\theta = 0.1$, $\delta = 0.9$
k	0.2416	0.2748	0.3645	0.4119	0.4431	0.4653
y	0.6530	0.6788	0.7388	0.7664	0.7833	0.7949
w	0.4571	0.4751	0.5172	0.5365	0.5483	0.5565
R	0.8108	0.7409	0.6080	0.5582	0.5304	0.5125
ca	0.2155	0.2003	0.1526	0.1246	0.1053	0.0911
co	0.1959	0.2036	0.2216	0.2299	0.2350	0.2385

Table 5: Determinant, trace and discriminant of matrix $\mathbf{J}^{SO}$ with $\alpha = 0.3$, $\beta = 0.9$ and $\gamma = 0.5$

θ	δ	$\det(\mathbf{J}^{SO})$	$\mathbf{T}_{\mathbf{J}}^{SO}$	Z
0.1	0.1	4	-17.04449	-13.04449
0.2	0.2	4	-7.140609	-3.140609
0.3	0.3	4	-3.905071	0.0949289
0.4	0.4	4	-2.337446	1.662553
0.5	0.5	4	-1.437521	2.562479
0.6	0.6	4	-0.8719509	3.128049
0.7	0.7	4	-0.4981759	3.501824
0.8	0.8	4	-0.2458125	3.754188
0.9	0.9	4	-0.0791088	3.920891
0.9	0.1	4	42.3286	46.3286
0.8	0.2	4	-0.112063	3.887937
0.7	0.3	4	-0.452936	3.547064
0.6	0.4	4	-0.8553743	3.144626
0.5	0.5	4	-1.437521	2.562479
0.4	0.6	4	-2.350999	1.649001
0.3	0.7	4	-3.93396	0.06604
0.2	0.8	4	-7.19476	-3.19476
0.1	0.9	4	-17.1709	-13.1709
0.1	0.9	4	-17.1709	-13.1709
0.2	0.8	4	-7.19476	-3.19476
0.3	0.7	4	-3.93396	0.06604
0.4	0.6	4	-2.350999	1.649001
0.5	0.5	4	-1.437521	2.562479
0.6	0.4	4	-0.8553743	3.144626
0.7	0.3	4	-0.452936	3.547064
0.8	0.2	4	-0.112063	3.887937
0.872863	**0.127137**	**4**	**0.7272058**	**4.727206**
0.9	0.1	4	42.3286	46.3286

Table 6: Determinant, trace and discriminant of matrix $\mathbf{J}^{SO}$ with $\alpha = 0.3$, $\beta = 0.9$ and $\gamma = 0.99$

θ	δ	$\det(\mathbf{J}^{SO})$	$\mathbf{T}_{\mathbf{J}}^{SO}$	Z
0.1	0.1	1.020304	-8.109779	-6.089577
0.2	0.2	1.020304	-3.117836	-1.097634
0.3	0.3	1.020304	-1.520497	0.4997047
0.4	0.4	1.020304	-0.7757596	1.244442
0.5	0.5	1.020304	-0.372435	1.647767
0.6	0.6	1.020304	-0.1395867	1.880615
0.7	0.7	1.020304	-0.0038876	2.016314
0.8	0.8	1.020304	0.0712562	2.091458
0.9	0.9	1.020304	0.1058284	2.12603
0.9	0.1	1.020304	0.3303701	2.350572
0.8	0.2	1.020304	0.1934865	2.213689
0.7	0.3	1.020304	0.064989	2.085191
0.6	0.4	1.020304	-0.1083589	1.911843
0.5	0.5	1.020304	-0.372435	1.647767
0.4	0.6	1.020304	-0.8052661	1.214936
0.3	0.7	1.020304	-1.582303	0.4378991
0.2	0.8	1.020304	-3.225467	-1.205265
0.1	0.9	1.020304	-8.339892	-6.319691
0.1	0.9	1.020304	-8.339892	-6.319691
0.2	0.8	1.020304	-3.225467	-1.205265
0.3	0.7	1.020304	-1.582303	0.4378991
0.4	0.6	1.020304	-0.8052661	1.214936
0.5	0.5	1.020304	-0.372435	1.647767
0.6	0.4	1.020304	-0.1083589	1.911843
0.7	0.3	1.020304	0.064989	2.085191
0.8	0.2	1.020304	0.1934865	2.213689
0.872863	**0.127137**	**1.020304**	**0.2863283**	**2.30653**
0.9	0.1	1.020304	0.3303701	2.350572

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